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# Stability and symmetry breaking in the general two-Higgs-doublet model

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**Abstract.** A method is presented for the analysis of the scalar potential in the general two-Higgs-doublet model. This allows us to give the conditions for the stability of the potential and for electroweak symmetry breaking in this model in a very concise way. These results are then applied to two different Higgs potentials in the literature, namely the MSSM and the two-Higgs-doublet potential proposed by Gunion et al. The known results for these models follow easily as special cases from the general results. In particular, for the potential of Gunion et al. we can clarify the stability and symmetry-breaking properties of the model with our method.

## 1 Introduction

The standard model (SM) of particle physics is theoretically consistent and experimentally successful to date [1, 2]. The recently observed neutrino masses are very small and can be neglected in most high-energy experiments. Only one ingredient of the SM, the Higgs boson, has yet to be discovered. From direct searches at LEP a lower bound on the Higgs boson mass of 114.4 GeV at 95% C.L. is obtained when the data from the four LEP collaborations are combined [3]. Furthermore, from measurements of electroweak precision observables at LEP, SLC and NuTeV that depend on the Higgs boson mass through radiative corrections and from other $W$-boson measurements, one obtains (see Table 10.2 in [1]) the prediction $m_H = 91^{+45}_{-32}\,\mathrm{GeV}$. Since the one-loop corrections depend on the Higgs mass only as $\log(m_H/m_W)$, the errors are rather large, and the upper error is larger than the lower one. Such an indirect determination of a particle mass is very successful in case of the top quark; see also Table 10.2 of [1]. However, there the observables have a quadratic dependence on the mass and are therefore much more predictive. A direct discovery of the Higgs boson at the LHC is presumably possible up to a mass of about 1 TeV; see e.g. [4]. One or several Higgs bosons may be found opening up the direct study of the scalar sector of particle physics.

Despite its experimental success, the SM is not satisfactory as a fundamental theory, not only because it contains a large number of parameters, that is coupling parameters and particle masses: the squared physical, renormalised Higgs boson mass $m_H^2$, which we expect to be of the order of the squared vacuum expectation value $v^2 \approx (250\,\mathrm{GeV})^2$ of the Higgs field, receives large quantum corrections. These corrections depend quadratically on the particle masses that the Higgs boson couples to. This means that the Higgs boson mass is sensitive to the heaviest particles of the theory, for instance from physics altering the high-energy behaviour at the GUT or the Planck scale. In principle, the corrections could be much larger than $v^2$ but cancel with the squared bare mass, so that the difference gives $m_H^2 \sim v^2$. However, such a fine-tuning is usually considered unnatural. For this so-called *naturalness problem*, see e.g. [5]. A systematic cancellation of quantum corrections to the squared Higgs boson mass is provided by supersymmetry [6, 7]. The simplest supersymmetric extension of the SM is the minimal supersymmetric standard model (MSSM) [8, 9], which has been studied extensively in the literature. In the MSSM one has two Higgs doublets. In further extensions like the next-to-minimal supersymmetric standard model (NMSSM) [10–12] a further Higgs singlet is added.

In this paper we study a class of general models having a scalar sector with two Higgs doublets. We suppose that the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry holds. In its simplest version the fermion content of such a model is assumed to be the same as that of the SM. The same is assumed for the gauge bosons, thereby avoiding the introduction of new fundamental interactions. In principle, electroweak symmetry breaking (EWSB) works in these models in a similar way as in the SM. The Lagrangian contains terms that consist only of scalar fields without derivatives. These terms form the scalar potential at tree level and are responsible for the stability and symmetry-breaking pattern of the model. Further, through

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their covariant derivatives the scalars have couplings to the gauge bosons, and through Yukawa interactions they couple to fermions. After EWSB these terms are responsible for the generation of the gauge-boson and fermion masses, respectively. With increasing number of scalar fields the number of parameters in the potential becomes soon very large. For instance, as we shall see in Sect. 3 below, there are 14 parameters to describe the most general potential with two Higgs doublets in contrast to only two parameters for one doublet. Therefore the characterisation of the symmetry breaking for different regions in parameter space becomes increasingly complicated. We present a formalism for the analysis of stability and of spontaneous symmetry breaking in models with two Higgs doublets.

There exists a vast amount of literature on the two-Higgs-doublet model (THDM), where typically the number of parameters of the potential is restricted by continuous or discrete symmetries. For instance in [13] a detailed discussion of the symmetry-breaking pattern for different regions in parameter space is given for the THDM where a $\mathbb{Z}_2$ symmetry is imposed on the Higgs potential and in [14] the stability of the *CP* conserving THDM is investigated.

We also want to mention an approach [15] to deduce the parameter constraints from stability and symmetry breaking in one specific model, the THDM introduced by Gunion et al. ([16, 17], see chapter 4). Further, in two complementary works [18–20], the hierarchy between the charge breaking and the charge conserving minima is investigated for the general THDM.

Basis independent techniques for the general THDM are used in different recent works [21–27] to analyse various aspects of the vacuum.

Here we deduce the parameter constraints from the stability and from the electroweak symmetry-breaking conditions in the general THDM. The global minimum of the potential is found by the determination of all stationary points. Our results agree with those of [13] if we impose the conditions on our parameters such that the potential is invariant under that discrete symmetry. Moreover, our formulation of the criteria for stability and EWSB of the potential is very concise and should, therefore, be interesting for its method. This general method, where the potential is expressed in terms of gauge invariant functions, was proposed already in a previous work [28].

We also remark that the scope of the present analysis is the *classical* level. In a more detailed study quantum corrections should be taken into account. Some aspects of radiative corrections for the Higgs potential in constrained $n$-Higgs-doublet models are discussed in [29]. The question if stability at the classical level is really necessary for a consistent quantum theory was put forward a long time ago by Symanzik [30]. The answer given in [31] was that, indeed, it is necessary. Thus, the results obtained at the classical level are important for the full theory.

This work is organised as follows: In Sect. 2 we give general motivations for an extended scalar sector and review theoretical and experimental constraints on Higgs boson masses in models with two Higgs doublets. In Sect. 3 we present the Lagrangian for the THDM. We introduce our notation for the Higgs potential, which is expressed in terms of gauge invariant functions of the fields. In Sect. 4 we analyse the conditions for the stability of the potential. In Sect. 5 we derive expressions for the location of the stationary points of the potential. The conditions derived from spontaneous symmetry breaking of the electroweak gauge group $SU(2)_L \times U(1)_Y$ down to the electromagnetic gauge group $U(1)_{\text{em}}$ are given in Sect. 6. In Sect. 7 we specify the potential after EWSB in our notation. Eventually, in Sect. 8 the results are applied to two specific models with two Higgs doublets, the MSSM and the model of Gunion et al. [16, 17]. We present our conclusions in Sect. 9.

In Appendix A and B we discuss the structure of the space of gauge orbits for the THDM and for the general model with arbitrary number of Higgs doublets, respectively.

## 2 Motivations for an extended Higgs sector

Given the fact that theoretically the mechanism of EWSB in the SM with one Higgs doublet is working well and that experimentally not even *one* fundamental scalar particle is discovered yet, what are the motivations to consider an extended Higgs sector? Some reasons are as follows.

A promising candidate for a theory that solves the naturalness problem and has a higher symmetry than the SM is the MSSM [8, 9]; for reviews, see e.g. [32, 33]. Also the MSSM contains fundamental Higgs fields that are responsible for the generation of masses. As the minimal supersymmetric extension of the SM, the MSSM has two scalar Higgs doublets, being the minimum for an analytic superpotential and the absence of triangle anomalies. An extended Higgs sector can improve gauge-coupling unification at high scales [34]. In particular, supersymmetric models allow the unification to occur at a sufficiently high scale consistent with the non-observation of proton decays [35–39]. We remark that supersymmetry imposes many relations between the parameters of the potential of the most general model with two doublets. Cosmology provides an additional reason for a non-minimal Higgs sector. The experimental lower bound on the Higgs boson mass in the SM, $m_H > 114.4\,\text{GeV}$, is too high for the electroweak phase transition in the early universe to provide the thermal instability that is necessary for baryogenesis [40]; for a review see [41]. In this respect models with additional scalar particles are more promising than the SM [41]. Last but not least, given the large spectrum of fermion masses and the fact that the fermion–scalar interactions are responsible for their generation, the idea does not seem too abstruse that several scalar particles are involved in this mechanism. There are three known generations of fermions so why should there exist only one Higgs boson?

A study of the general Higgs sector of a theory possessing the gauge group $SU(3)_C \times SU(2)_{\text{L}} \times U(1)_Y$ was presented in [42]. In the following the strong interaction gauge group $SU(3)_C$ will play no role, so we shall not mention it further. In [42] the scalar fields, collectively denoted by $\chi$, are supposed to transform under a general representation of the gauge group $SU(2)_L \times U(1)_Y$. Such a representation may be reducible and consists of complex unitary and real orthogonal parts. However one can show that without loss of generality it can be assumed that $\chi$ carries a real orthogonal representation of $SU(2)_L \times U(1)_Y$ [42]. For the THDM this correspondence is demonstrated in Appendix B of [43]. The scalar potential is then assumed in [42] to have a non-zero vacuum expectation value,

$$\mathbf{v} \equiv \langle 0|\chi|0\rangle \neq 0, \tag{1}$$

and to leave the electromagnetic subgroup $U(1)_{\text{em}}$ unbroken as usual. We use the boldface letter here in order to signify that $\mathbf{v}$ is, like $\chi$, in general a multi-component vector. One can then compute particle masses and couplings for arbitrary representations of scalars. However, only some representations are allowed in order to be in agreement with experimental data.

One main restriction originates from the observed high suppression of flavour-changing neutral currents. A way to ensure this in the theory is to require that all quarks of a given charge receive their masses from the vacuum expectation value of the same Higgs boson [44]. Since we analyse only the scalar potential in this work and do not specify the Yukawa interactions, we shall not discuss this condition further here.

Very relevant for us here are the consequences for the Higgs sector obtained from the accurately measured $\rho$-parameter, which relates the masses of the $W$ and $Z$ bosons, $m_W$ and $m_Z$, to the weak mixing angle $\theta_{\text{W}}$:

$$\rho \equiv \left(\frac{m_W}{\cos\theta_{\text{W}}\, m_Z}\right)^2 . \tag{2}$$

Experimentally the $\rho$-parameter is very close to 1 [45], and this suggests to require theoretically $\rho = 1$ at tree level. This is indeed the case for the SM. For the most general Higgs model as studied in [42] one finds the following.

It is convenient to extend the real representation carried by the general Higgs field $\chi$ to a unitary representation of the same (complex) dimension and to decompose it into representations with definite values $(t, y)$, where $t$ and $y$ are the weak-isospin and weak-hypercharge quantum numbers, respectively. We have

$$t = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \dots \tag{3}$$

and, for reasons discussed in [42], suppose the hypercharge quantum numbers $y$ to be rational numbers. The normalisation is such that the charge, hypercharge and third component of weak-isospin matrices are related by

$$Q = T_3 + Y\,. \tag{4}$$

Then the squared gauge-boson masses (see (2.43) in [42]) are given by

$$m_W^2 = \frac{1}{2}\left(\frac{e}{\sin\theta_{\text{W}}}\right)^2 \sum_{t,y} \left[t(t+1) - y^2\right] \mathbf{v}^{\text{T}}\mathbb{P}(t,y)\mathbf{v}, \tag{5}$$

$$m_Z^2 = \left(\frac{e}{\sin\theta_{\text{W}}\cos\theta_{\text{W}}}\right)^2 \sum_{t,y} y^2\, \mathbf{v}^{\text{T}}\mathbb{P}(t,y)\mathbf{v}\,, \tag{6}$$

where $\mathbb{P}(t, y)$ is the projector on the subspace with representation $(t, y)$. Here, the positron charge $e$, and the sine and cosine of the weak mixing angle are defined in terms of the gauge couplings $g$ and $g'$ as in the SM (see for instance [46]):

$$\sin\theta_{\text{W}} = \frac{g'}{\sqrt{g^2 + g'^2}}\,, \tag{7}$$

$$\cos\theta_{\text{W}} = \frac{g}{\sqrt{g^2 + g'^2}}\,, \tag{8}$$

$$e = g\sin\theta_{\text{W}}\,, \tag{9}$$

where we use the same notation as in [42]. It is shown in [42] that

$$\mathbf{v}^{\text{T}}\mathbb{P}(t,y)\mathbf{v} \neq 0 \tag{10}$$

is only possible if

$$y \in \{-t, -t+1, \dots, t\}. \tag{11}$$

Inserting the expressions for $m_W$ and $m_Z$ in the definition (2) one obtains [42]

$$\rho = \frac{\sum_{t,y}[t(t+1) - y^2]\,\mathbf{v}^{\text{T}}\mathbb{P}(t,y)\mathbf{v}}{\sum_{t,y} 2y^2\,\mathbf{v}^{\text{T}}\mathbb{P}(t,y)\mathbf{v}}\,. \tag{12}$$

To obtain $\rho = 1$ one can either fine-tune the parameters of the potential in order to get the right vacuum expectation values, which seems rather unnatural and is therefore not considered here. Or one can only allow those representations in (5) and (6) that *separately* lead to $\rho = 1$. There are infinitely many such representations [47], starting with the doublet with $t = 1/2$ and $y = \pm 1/2$, and the septuplet with $t = 3$ and $y = \pm 2$. From each of these representations one or more copies are allowed and one still gets $\rho = 1$. Furthermore, the singlet with $y = 0$ and all representations with

$$y \notin \{-t, -t+1, \dots, t\} \tag{13}$$

can occur because they do not contribute to the sums in (12).

The simplest possibility to extend the Higgs sector of the SM keeping $\rho = 1$ at tree level is, therefore, to allow for more than one Higgs doublet. In these models the shape of the scalar potential depends on many parameters and can be quite complicated. As mentioned above it is the potential that is responsible for the scalar self-interactions and – together with the interaction terms of the scalars with the respective particle – for the generation of the masses. Therefore one is interested in the conditions that one has to impose on these parameters in order to render the potential stable and to guarantee spontaneous symmetry breaking from $SU(2)_L \times U(1)_Y$ to $U(1)_{\text{em}}$. In this paper we consider two Higgs doublets; that is, the THDM. We remark that after EWSB three degrees of freedom in the scalar sector reappear as longitudinal modes of the massive gauge-bosons. All other degrees of freedom of the scalar sector correspond to physical Higgs bosons, that is with each additional doublet four (real) physical scalar degrees of freedom are added to the model. In the THDM, there are altogether five physical Higgs particles: three neutral Higgs bosons $h^0$, $H^0$ (where conventionally $m_{h^0} \le m_{H^0}$) and $A^0$, as well as two charged Higgs bosons $H^\pm$. If the Higgs potential is $CP$ conserving the neutral mass eigenstates are also $CP$ eigenstates, where $h^0$ and $H^0$ are scalar bosons and $A^0$ is a pseudoscalar. There exist various studies of the phenomenology of the THDM in the literature; for an overview and further references see for instance [48].

For the MSSM a large number of Feynman rules involving Higgs bosons is derived in [49, 50]. The phenomenology of the Higgs bosons in the MSSM is further developed in [51–54]. We remark that in models that possess an extended Higgs sector (and may also contain further non-SM particles) for certain regions of the parameter space there often exists one neutral Higgs boson that behaves similarly to the SM Higgs boson. For instance, the MSSM Higgs sector is described by two parameters, which can be chosen as the mass of the pseudoscalar boson $m_{A^0}$ and the ratio $\tan\beta$ of the vacuum expectation values of the two Higgs doublets. In the decoupling limit $m_{A^0} \gg m_Z$, where practically $m_{A^0} \gtrsim 200$ GeV is sufficient [55], one neutral Higgs boson $h^0$ is light and has the same couplings as the SM Higgs boson, whereas the other Higgs bosons $H^0$, $A^0$ and $H^\pm$ are heavy and decouple. If there exist light supersymmetric particles that couple to $h^0$ it may be comparatively easy to distinguish $h^0$ from the SM Higgs boson even if it has SM-like couplings; this is because $h^0$ can decay into the light supersymmetric particles if kinematically allowed. Further, if the light supersymmetric particles couple to photons (gluons) the one-loop $\gamma\gamma h^0$ ($ggh^0$) coupling is modified by their contribution to the loop, thus the branching ratios of $h^0$ differ from those of the SM Higgs boson. If all heavy Higgs bosons are beyond kinematical reach in the decoupling limit, such precision measurements are the only way to distinguish $h^0$ from the SM Higgs boson. Notice that at an $e^+e^-$ collider like the ILC [56–58] the heavy Higgs states can only be produced pairwise in the decoupling limit, so that the kinematical limit may be very crucial. However, at a $\gamma\gamma$ collider $s$-channel resonant $H^0$ and $A^0$ production [59] is possible, so that only the available c.m. energy of the $\gamma\gamma$ system limits the masses which can be explored. In the $\gamma\gamma$ option of an ILC one expects that the maximal useful $\gamma\gamma$ c.m. energy will be 80% of the c.m. energy in the $e^+e^-$ mode [60].

Present experiments give the following exclusion regions for various versions of the THDM. The OPAL collaboration has performed a parameter scan for the $CP$ conserving THDM [61] and excluded at 95% C.L. large parts of the region where

$$\begin{aligned} &1\,\text{GeV} \le m_{h^0} \le 130\,\text{GeV}\,,\\ &3\,\text{GeV} \le m_{A^0} \le 2\,\text{TeV}\,,\\ &0.4 \le \tan\beta \le 40\,,\\ &\alpha = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}\,. \end{aligned} \tag{14}$$

Here $\alpha$ is a mixing angle for the two states $h^0$ and $H^0$. Further, the approximate region where

$$\begin{aligned} &1\,\text{GeV} < m_{h^0} < 55\,\text{GeV}\,,\\ &3\,\text{GeV} < m_{A^0} < 63\,\text{GeV} \end{aligned} \tag{15}$$

is excluded for all $\tan\beta$ values for negative $\alpha$. In a combined analysis [62] of the four LEP collaborations a lower bound on the mass of the charged Higgs in models with two Higgs doublets like the THDM or the MSSM, we have approximately

$$m_{H^\pm} > 78.6\,\text{GeV} \tag{16}$$

is determined. In another analysis [63] of the four LEP collaborations, signals for neutral Higgs bosons at different benchmark points of the MSSM were searched for. Here the limits

$$\begin{aligned} &m_{h^0} > 91.0\,\text{GeV}\,,\\ &m_{A^0} > 91.9\,\text{GeV} \end{aligned} \tag{17}$$

at 95% C.L. are obtained. Under the assumption that the "left–right"–stop mixing is maximal and with *conservative* choices for other MSSM parameters the region $0.5 < \tan\beta < 2.4$ is excluded at 95% C.L.

## 3 The general two-Higgs-doublet model

We denote the two complex Higgs-doublet fields by

$$\varphi_i(x) = \begin{pmatrix} \varphi_i^+(x) \\ \varphi_i^0(x) \end{pmatrix}, \tag{18}$$

with $i = 1, 2$. Hence we have eight real scalar degrees of freedom. The most general $SU(2)_L \times U(1)_Y$ invariant Lagrangian for the THDM can be written as

$$\mathcal{L}_{\text{THDM}} = \mathcal{L}_\varphi + \mathcal{L}_{\text{Yuk}} + \mathcal{L}'\,, \tag{19}$$

where the pure Higgs boson Lagrangian is given by

$$\mathcal{L}_\varphi = \sum_{i=1,2} (\mathcal{D}_\mu \varphi_i)^\dagger (\mathcal{D}^\mu \varphi_i) - V(\varphi_1, \varphi_2)\,. \tag{20}$$

This term replaces the kinetic terms of the Higgs boson and the Higgs potential in the SM Lagrangian. The covariant derivative is

$$\mathcal{D}_\mu = \partial_\mu + \mathrm{i} g W_\mu^a \mathbf{T}_a + \mathrm{i} g' B_\mu \mathbf{Y}\,, \tag{21}$$

where $\mathbf{T}_a$ and $\mathbf{Y}$ are the generating operators of the weak-isospin and weak-hypercharge transformations. For the Higgs doublets we have $\mathbf{T}_a = \tau_a/2$, where $\tau_a$ $(a=1,2,3)$ are the Pauli matrices. We assume both doublets to have weak hypercharge $y = 1/2$. Further, $\mathcal{L}_{\rm Yuk}$ are the Yukawa-interaction terms of the Higgs fields with fermions. Finally, $\mathcal{L}'$ contains the terms of the Lagrangian without Higgs fields. We do not specify $\mathcal{L}_{\rm Yuk}$ and $\mathcal{L}'$ here since they are not relevant for our analysis. The Higgs potential $V$ in the THDM will be specified below and discussed extensively.

We remark that in the MSSM the two Higgs doublets $H_1$ and $H_2$ carry hypercharges $y=-1/2$ and $y=+1/2$, respectively, whereas here we use the conventional definition of the THDM with both doublets carrying $y=+1/2$. However, our analysis can be translated to the other case, see for example (3.1) in [49, 50], by setting

$$\begin{aligned}\varphi_1^\alpha &= -\epsilon_{\alpha\beta}\left(H_1^\beta\right)^*,\\ \varphi_2^\alpha &= H_2^\alpha,\end{aligned}\tag{22}$$

where $\epsilon$ is given by

$$\epsilon = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.\tag{23}$$

The most general gauge invariant and renormalisable potential $V(\varphi_1,\varphi_2)$ for the two Higgs doublets $\varphi_1$ and $\varphi_2$ is a hermitian linear combination of the following terms:

$$\varphi_i^\dagger\varphi_j, \quad (\varphi_i^\dagger\varphi_j)(\varphi_k^\dagger\varphi_l),\tag{24}$$

where $i,j,k,l \in \{1,2\}$. It is convenient to discuss the properties of the potential such as its stability and its spontaneous symmetry breaking in terms of gauge invariant expressions. For this purpose we arrange the $SU(2)_L\times U(1)_Y$ invariant scalar products into the hermitian $2\times 2$ matrix

$$\underline{K} := \begin{pmatrix}\varphi_1^\dagger\varphi_1 & \varphi_2^\dagger\varphi_1\\ \varphi_1^\dagger\varphi_2 & \varphi_2^\dagger\varphi_2\end{pmatrix}\tag{25}$$

and consider its decomposition

$$\underline{K}_{ij} = \frac{1}{2}\left(K_0\,\delta_{ij} + K_a\,\sigma^a_{ij}\right),\tag{26}$$

using the completeness of the Pauli matrices $\sigma^a$ $(a=1,2,3)$, together with the unit matrix. The four real coefficients defined by the decomposition (26) are given by

$$K_0 = \varphi_i^\dagger\varphi_i\,,\qquad K_a = \left(\varphi_i^\dagger\varphi_j\right)\sigma^a_{ij}\quad (a=1,2,3).\tag{27}$$

Here and in the following summation over repeated indices is understood. Using the inversion of (27),

$$\begin{aligned}\varphi_1^\dagger\varphi_1 &= (K_0+K_3)/2\,, & \varphi_1^\dagger\varphi_2 &= (K_1+\mathrm{i}K_2)/2\,,\\ \varphi_2^\dagger\varphi_2 &= (K_0-K_3)/2\,, & \varphi_2^\dagger\varphi_1 &= (K_1-\mathrm{i}K_2)/2\,,\end{aligned}\tag{28}$$

the most general potential can be written as follows:

$$V(\varphi_1,\varphi_2) = V_2 + V_4\,,\tag{29a}$$
$$V_2 = \xi_0 K_0 + \xi_a\,K_a\,,\tag{29b}$$
$$V_4 = \eta_{00}K_0^2 + 2K_0\eta_a\,K_a + K_a\eta_{ab}K_b\,,\tag{29c}$$

where the 14 independent parameters $\xi_0$, $\xi_a$, $\eta_{00}$, $\eta_a$ and $\eta_{ab}=\eta_{ba}$ are real. We subsequently write $\mathbf{K} := (K_a)$, $\boldsymbol{\xi} := (\xi_a)$, $\boldsymbol{\eta} := (\eta_a)$ and $E := (\eta_{ab})$.

Now we consider a change of basis of the Higgs fields, $\varphi_i \to \varphi_i'$, where

$$\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}.\tag{30}$$

Here

$$U = \begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\qquad (U^\dagger U = \mathbb{1})\tag{31}$$

is a $2\times 2$ unitary transformation. With (30) the gauge invariant functions (27) transform as

$$K_0' = K_0\,,\qquad K_a' = R_{ab}(U)K_b\,,\tag{32}$$

where $R_{ab}(U)$ is defined by

$$U^\dagger\sigma^a\,U = R_{ab}(U)\,\sigma^b\,.\tag{33}$$

The matrix $R(U)$ has the properties

$$R^*(U) = R(U)\,,\quad R^{\mathrm{T}}(U)\,R(U) = \mathbb{1}\,,\quad \det R(U) = 1\,,\tag{34}$$

where $\mathbb{1}$ denotes the $3\times 3$ unit matrix. That is, $R(U)\in SO(3)$. The form of the Higgs potential (29) remains unchanged under the replacement (32) if we perform an appropriate transformation of the parameters

$$\begin{aligned}\xi_0' &= \xi_0\,,\ \boldsymbol{\xi}' = R(U)\,\boldsymbol{\xi}\,,\\ \eta_{00}' &= \eta_{00}\,,\ \boldsymbol{\eta}' = R(U)\,\boldsymbol{\eta}\,,\\ E' &= R(U)\,E\,R^{\mathrm{T}}(U)\,.\end{aligned}\tag{35}$$

Moreover, for every matrix $R$ with the properties (34), there is a unitary transformation (30). We can therefore diagonalise $E$, thereby reducing the number of parameters of $V$ by three. The Higgs potential is then determined by only 11 real parameters.

The matrix $\underline{K}$ is positive semi-definite, which follows immediately from its definition (25). With $K_0 = \mathrm{tr}\,\underline{K}$ and $K_0^2 - \mathbf{K}^2 = 4\det\underline{K}$ this implies

$$K_0 \ge 0\,,\qquad K_0^2 - \mathbf{K}^2 \ge 0\,.\tag{36}$$

On the other hand, for any given $K_0, \mathbf{K}$ fulfilling (36), it is possible to find fields $\varphi_i$ obeying (27). Furthermore, all fields obeying (27) for a given $K_0, \mathbf{K}$ form one gauge orbit. This is shown explicitly in Appendix A.

Thus, the functions $K_0, K_a$ parametrise the gauge orbits and not a unique Higgs-field configuration. Specifying the domain of the functions $K_0, K_a$ corresponding to the gauge orbits allows one to discuss the potential directly in the form (29) with all gauge degrees of freedom eliminated. It is curious to note that the gauge orbits of the Higgs fields of the THDM are parametrised by Minkowski-type four vectors $(K_0, \mathbf{K})$ which have to lie on or inside the forward light cone.

In the following sections we derive bounds on the parameters of the potential that result from the conditions that

– the potential $V$ is stable,
– we have spontaneous symmetry breaking of $SU(2)_{\mathrm{L}} \times U(1)_Y$ down to $U(1)_{\mathrm{em}}$ .

## 4 Stability

According to Sect. 3 we can analyse the properties of the potential (29) as a function of $K_0$ and $\mathbf{K}$ on the domain determined by $K_0 \geq 0$ and $K_0^2 \geq \mathbf{K}^2$. For $K_0 > 0$ we define

$$\mathbf{k} := \mathbf{K}/K_0 \,. \tag{37}$$

In fact, we have $K_0 = 0$ only for $\varphi_1 = \varphi_2 = 0$, and the potential is $V = 0$ in this case. From (29) and (37) we obtain for $K_0 > 0$

$$V_2 = K_0\, J_2(\mathbf{k}), \quad J_2(\mathbf{k}) := \xi_0 + \boldsymbol{\xi}^{\mathrm{T}}\mathbf{k}\,, \tag{38}$$
$$V_4 = K_0^2\, J_4(\mathbf{k})\,, \quad J_4(\mathbf{k}) := \eta_{00} + 2\boldsymbol{\eta}^{\mathrm{T}}\mathbf{k} + \mathbf{k}^{\mathrm{T}}E\mathbf{k}\,, \tag{39}$$

where we introduce the functions $J_2(\mathbf{k})$ and $J_4(\mathbf{k})$ on the domain $|\mathbf{k}| \leq 1$.

For the potential to be stable, it must be bounded from below. The stability is determined by the behaviour of $V$ in the limit $K_0 \to \infty$; hence by the signs of $J_4(\mathbf{k})$ and $J_2(\mathbf{k})$ in (38) and (39). For the theory to be at least *marginally* stable

$$\left.\begin{array}{ll} J_4(\mathbf{k}) > 0 & \text{or} \\ J_4(\mathbf{k}) = 0 & \text{and} \quad J_2(\mathbf{k}) \geq 0 \end{array}\right\} \text{ for all } |\mathbf{k}| \leq 1 \tag{40}$$

is necessary and sufficient, since this condition is equivalent to $V \geq 0$ for $K_0 \to \infty$ in all possible directions $\mathbf{k}$. The more robust stability property $V \to \infty$ for $K_0 \to \infty$ and any $\mathbf{k}$ can either be guaranteed by

$$\left.\begin{array}{ll} J_4(\mathbf{k}) > 0 & \text{or} \\ J_4(\mathbf{k}) = 0 & \text{and} \quad J_2(\mathbf{k}) > 0 \end{array}\right\} \text{ for all } |\mathbf{k}| \leq 1 \tag{41}$$

in a *weak* sense, or by

$$J_4(\mathbf{k}) > 0 \quad \text{for all } |\mathbf{k}| \leq 1\,, \tag{42}$$

in a *strong* sense; that is, by the quartic terms of $V$ solely.

To assure $J_4(\mathbf{k})$ is positive (semi-) definite, it is sufficient to consider its value for all stationary points of $J_4(\mathbf{k})$ on the domain $|\mathbf{k}| < 1$, and for all stationary points on the boundary $|\mathbf{k}| = 1$. This holds, because the global minimum of the continuous function $J_4(\mathbf{k})$ is reached on the compact domain $|\mathbf{k}| \leq 1$, and it is among those stationary points. This leads to bounds on $\eta_{00}$, $\eta_a$ and $\eta_{ab}$, which parametrise the quartic term $V_4$ of the potential. For $|\mathbf{k}| < 1$ the stationary points – if there are any – must fulfil

$$E\mathbf{k} = -\boldsymbol{\eta} \quad \text{with } |\mathbf{k}| < 1\,. \tag{43}$$

If $\det E \neq 0$ we explicitly obtain

$$J_4(\mathbf{k})\big|_{\mathrm{stat}} = \eta_{00} - \boldsymbol{\eta}^{\mathrm{T}} E^{-1} \boldsymbol{\eta} \quad \text{if } 1 - \boldsymbol{\eta}^{\mathrm{T}} E^{-2} \boldsymbol{\eta} > 0\,, \tag{44}$$

where the inequality follows from the condition $|\mathbf{k}| < 1$. If $\det E = 0$ there can exist one or more "exceptional" solutions $\mathbf{k}$ of (43). They, again, have to obey $|\mathbf{k}| < 1$. For $|\mathbf{k}| = 1$ we must find the stationary points of the function

$$F_4(\mathbf{k}, u) := J_4(\mathbf{k}) + u\left(1 - \mathbf{k}^2\right)\,, \tag{45}$$

where $u$ is a Lagrange multiplier. Those are given by

$$(E - u)\mathbf{k} = -\boldsymbol{\eta} \quad \text{with } |\mathbf{k}| = 1\,. \tag{46}$$

For regular values of $u$ such that $\det(E - u) \neq 0$ the stationary points are given by

$$\mathbf{k}(u) = -(E - u)^{-1}\boldsymbol{\eta}\,, \tag{47}$$

and the Lagrange multiplier is determined from the condition $\mathbf{k}^{\mathrm{T}}\mathbf{k} = 1$ after inserting (47):

$$1 - \boldsymbol{\eta}^{\mathrm{T}}(E - u)^{-2}\boldsymbol{\eta} = 0\,. \tag{48}$$

We thus obtain the solution

$$J_4(\mathbf{k})\big|_{\mathrm{stat}} = u + \eta_{00} - \boldsymbol{\eta}^{\mathrm{T}}(E - u)^{-1}\boldsymbol{\eta}\,, \tag{49}$$

where $u$ is a solution of (48). Also for $|\mathbf{k}| = 1$, depending on the parameters $\eta_a$ and $\eta_{ab}$, there can be exceptional solutions $(\mathbf{k}, u)$ of (46) where $\det(E - u) = 0$, i.e. where $u$ is an eigenvalue of $E$.

The regular solutions for the two cases $|\mathbf{k}| < 1$ and $|\mathbf{k}| = 1$ can be described using one function only. Considering (45) and (47), we define

$$f(u) := F_4(\mathbf{k}(u), u)\,, \tag{50}$$

with $\mathbf{k}(u)$ as in (47). This leads to

$$f(u) = u + \eta_{00} - \boldsymbol{\eta}^{\mathrm{T}}(E - u)^{-1}\boldsymbol{\eta}\,, \tag{51}$$
$$f'(u) = 1 - \boldsymbol{\eta}^{\mathrm{T}}(E - u)^{-2}\boldsymbol{\eta}\,, \tag{52}$$

so that for all "regular" stationary points $\mathbf{k}$ of $J_4(\mathbf{k})$

$$f(u) = J_4(\mathbf{k})\big|_{\mathrm{stat}}\,, \tag{53}$$
$$f'(u) = 1 - \mathbf{k}^2 \tag{54}$$

holds, where we set $u = 0$ for the solution with $|\mathbf{k}| < 1$. There are stationary points of $J_4(\mathbf{k})$ with $|\mathbf{k}| < 1$ and $|\mathbf{k}| = 1$ exactly if $f'(0) > 0$ and $f'(u) = 0$, respectively, and the value of $J_4(\mathbf{k})$ is then given by $f(u)$.

In a basis where $E = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ we obtain

$$f(u) = u + \eta_{00} - \sum_{a=1}^{3} \frac{\eta_a^2}{\mu_a - u}\,, \tag{55}$$

$$f'(u) = 1 - \sum_{a=1}^{3} \frac{\eta_a^2}{(\mu_a - u)^2}\,. \tag{56}$$

The derivative $f'(u)$ has at most six zeros. The shape of $f(u)$ and $f'(u)$ for a (purely didactical) set of parameters where $f'(u)$ has six zeros can be seen in Fig. 1. Notice that there are no exceptional solutions if in this basis all three components of $\boldsymbol{\eta}$ are different from zero.

The function $f(u)$ given by (51) allows us to discuss also the exceptional solutions of (43) and (46). Consider first $|\mathbf{k}| < 1$ and suppose that $\det E = 0$. In the basis where $E$ is diagonal we have

$$\det E = \mu_1\, \mu_2\, \mu_3 = 0\,, \tag{57}$$

and (43) reads

$$\begin{aligned} \mu_1\, k_1 &= -\eta_1\,,\\ \mu_2\, k_2 &= -\eta_2\,,\\ \mu_3\, k_3 &= -\eta_3\,. \end{aligned} \tag{58}$$

Clearly, a solution of (58) is only possible if with $\mu_a = 0$ also $\eta_a = 0$ $(a = 1, 2, 3)$. Therefore, we see from (55) that

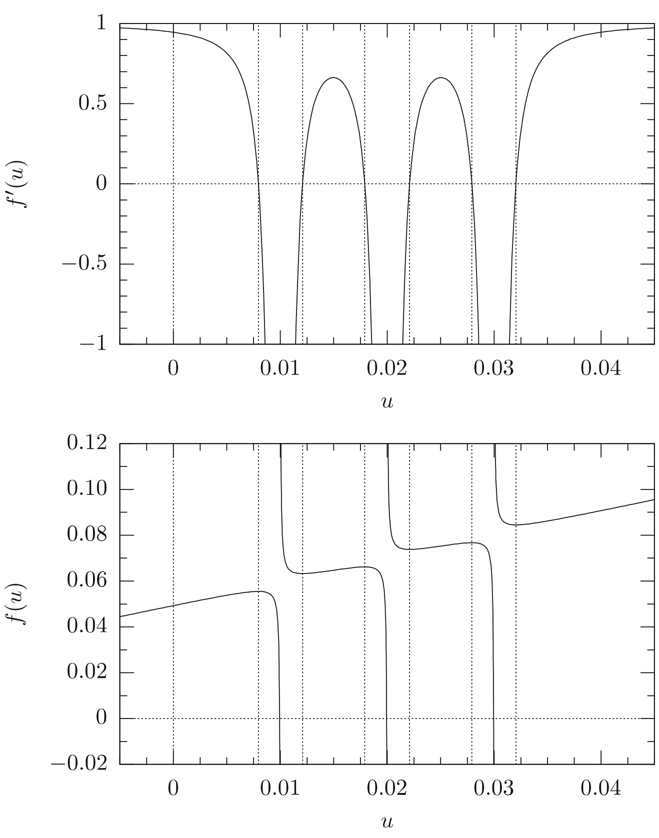


**Fig. 1.** The stability determining functions $f'(u)$ and $f(u)$ as given by (56) and (55) with $\eta_{00} = 0.05$, $(\mu_1, \mu_2, \mu_3) = (0.01, 0.02, 0.03)$ and $(\eta_1, \eta_2, \eta_3) = (0.002, 0.002, 0.002)$

exceptional solutions with $|\mathbf{k}| < 1$ are only possible if $f(u)$ stays finite at $u = 0$. That is, the pole which would correspond to $\mu_a = 0$ must have residue zero. Suppose now that indeed $\eta_a = 0$ for all $a$ where $\mu_a = 0$. Take as an example $\mu_1 = \mu_2 = 0$ and $\eta_1 = \eta_2 = 0$, but $\mu_3 \neq 0$. Then we get the general solution of (58) as follows:

$$k_3 = -\frac{\eta_3}{\mu_3}\,, \tag{59}$$

with $k_1$, $k_2$ arbitrary but satisfying

$$\mathbf{k}^2 = k_1^2 + k_2^2 + \left(\frac{\eta_3}{\mu_3}\right)^2 < 1\,. \tag{60}$$

We can write this as

$$\mathbf{k} = \mathbf{k}_{\parallel} + \mathbf{k}_{\perp}\,, \tag{61}$$

where

$$\begin{gathered} \mathbf{k}_{\parallel} = -\frac{1}{\mu_3}\boldsymbol{\eta}\,, \qquad E\,\mathbf{k}_{\perp} = 0\,,\\ \mathbf{k}_{\perp}^2 < 1 - \mathbf{k}_{\parallel}^2 = 1 - \left(\frac{\eta_3}{\mu_3}\right)^2\,. \end{gathered} \tag{62}$$

For the functions (55) and (56) we get here

$$f(u) = u + \eta_{00} - \frac{\eta_3^2}{\mu_3 - u}\,, \tag{63}$$

$$f'(u) = 1 - \frac{\eta_3^2}{(\mu_3 - u)^2}\,. \tag{64}$$

Inserting the solution $\mathbf{k}$ from (61) and (62) in $J_4(\mathbf{k})$ we get

$$f(0) = \left. J_4(\mathbf{k})\right|_{\mathrm{stat}}\,, \tag{65}$$

$$f'(0) = 1 - \mathbf{k}_{\parallel}^2 > \mathbf{k}_{\perp}^2 \geq 0\,. \tag{66}$$

Clearly these arguments work similarly, if only one of the $\mu_a$ is equal to zero or all three $\mu_a$ are zero. In all cases (65) holds for the exceptional points with $|\mathbf{k}| < 1$, which can exist only if $f(u)$ has no pole at $u = 0$. Since (65) involves only "scalar" quantities, it holds in any basis.

The case of exceptional solutions for $|\mathbf{k}| = 1$ can be treated in an analogous way. An exceptional solution of (46) with $u = \mu_a$ $(a = 1, 2, 3)$ can only exist if the corresponding $\eta_a = 0$. Then $f(u)$ has no pole for $u = \mu_a$ and the exceptional solutions of (46) fulfil

$$\mathbf{k} = \mathbf{k}_{\parallel} + \mathbf{k}_{\perp}\,, \tag{67}$$

with

$$\mathbf{k}_{\parallel} = -\left.(E - u)^{-1}\boldsymbol{\eta}\right|_{u=\mu_a}\,, \quad (E - \mu_a)\mathbf{k}_{\perp} = 0$$

and

$$f(\mu_a) = \left. J_4(\mathbf{k})\right|_{\mathrm{stat}}\,, \tag{68}$$

$$f'(\mu_a) = 1 - \mathbf{k}_{\parallel}^2 = \mathbf{k}_{\perp}^2 \geq 0\,. \tag{69}$$

Note that if a solution is possible, $\mathbf{k}_\perp$ may be any linear combination of the eigenvectors to the eigenvalue $\mu_a$ of $E$, where the norm is given by $|\mathbf{k}_\perp| = \sqrt{f'(\mu_a)}$. Thus we see that the function $f(u)$ is very useful for discussing the stability of the THDM potential. What we have shown so far can be formulated as follows.

Consider the functions $f(u)$ and $f'(u)$. Denote by $I$,

$$I = \{u_1, \dots, u_n\}\,, \tag{70}$$

the following set of values of $u$. Include in $I$ all $u$ where $f'(u) = 0$. Add $u = 0$ to $I$ if $f'(0) > 0$. Consider then the eigenvalues $\mu_a$ $(a = 1,2,3)$ of $E$. Add those $\mu_a$ to $I$ where $f(\mu_a)$ is finite and $f'(\mu_a) \geq 0$. We have $n \leq 10$. The values of the function $J_4(\mathbf{k})$ at its stationary points are given by

$$J_4(\mathbf{k})|_{\text{stat}} = f(u_i)\,, \tag{71}$$

with $u_i \in I$. The potential is stable if $f(u_i) > 0$ for all $u_i \in I$. Then the stability is given solely by the quartic terms in the potential. The potential is unstable if we have $f(u_i) < 0$ for at least one $u_i \in I$. If we have $f(u_i) \geq 0$ for all $u_i \in I$ and $f(u_i) = 0$ for at least one $u_i \in I$, we have to consider in addition $J_2(\mathbf{k})$ in order to decide on the stability of the potential.

We turn now to this latter case. We shall show that we have then to consider in addition the function

$$g(u) := \xi_0 - \boldsymbol{\xi}^{\mathrm{T}}(E-u)^{-1}\boldsymbol{\eta}\,. \tag{72}$$

For the stationary points of $J_4(\mathbf{k})$ with

$$J_4(\mathbf{k})|_{\text{stat}} = f(u_i) = 0\,. \tag{73}$$

We have for the vectors $\mathbf{k}$ satisfying (73)

$$J_2(\mathbf{k}) = g(u_i)\,, \tag{74}$$

if $u_i \neq \mu_a$; that is, $u_i$ is not an eigenvalue of $E$. If $u_i$ is an eigenvalue of $E$; that is, $u_i = \mu_a \in I$, and $\mathbf{e}_l(u_i)$ $(l = 1, \dots, N)$ are the $N \leq 3$ eigenvectors to $u_i$, then we have

$$\inf_{\mathbf{k}} J_2(\mathbf{k}) = g(u_i) - |\boldsymbol{\xi}_\perp(u_i)|\,\sqrt{f'(u_i)}\,, \tag{75}$$

where the infimum is taken over all exceptional solutions $\mathbf{k}$ to $u_i$ and

$$\boldsymbol{\xi}_\perp(u_i) := \sum_{l=1}^{N} \frac{\boldsymbol{\xi}\mathbf{e}_l(u_i)}{|\mathbf{e}_l(u_i)|^2}\,\mathbf{e}_l(u_i)\,. \tag{76}$$

We summarise our findings in a theorem.

**Theorem 1.** *The most general potential of the two-Higgs-doublet model has the form* (29)*. Its stability is decided in the following way. If the potential has only the quadratic term* $V_2$*, it is stable for* $\xi_0 > |\boldsymbol{\xi}|$*, marginally stable for* $\xi_0 = |\boldsymbol{\xi}|$ *and unstable for* $\xi_0 < |\boldsymbol{\xi}|$*. Suppose now that* $V_4 \not\equiv 0$*. We construct then the functions* $f(u)$ *of* (51)*,* $f'(u)$ *of* (52) *and* $g(u)$ *of* (72)*, and the set* $I$ (70) *of (at most 10)* $u$ *values.*

1. *If* $f(u_i) > 0$ *for all* $u_i \in I$*, the potential is stable in the strong sense* (42)*.*
2. *If* $f(u_i) < 0$ *for at least one* $u_i \in I$*, the potential is unstable.*
3. *If* $f(u_i) \geq 0$ *for all* $u_i \in I$ *and* $f(u_i) = 0$ *for at least one* $u_i \in I$*, we consider also the function* $g(u)$ (72)*. The potential is stable in the weak sense* (41) *if for all* $u_i \in I$ *where* $f(u_i) = 0$ *the following holds (see* (74) *to* (76)*):*

$$g(u_i) > 0 \text{ if } u_i \neq \mu_a\,, \tag{77}$$

$$g(u_i) - |\boldsymbol{\xi}_\perp(u_i)|\,\sqrt{f'(u_i)} > 0 \text{ if } u_i = \mu_a\,. \tag{78}$$

*If in some or all of these cases we have* $= 0$ *instead of* $> 0$ *we have marginal stability* (40)*. If in at least one case we have* $< 0$ *instead of* $> 0$*, the potential is unstable.*

Our theorem gives a complete characterisation of the stability properties of the general THDM potential. In the following subsection we apply the theorem to assert that the strong stability condition (42) holds for a specific potential. An application for the weaker stability condition (41) is given in Sect. 8.1.

### 4.1 Stability for THDM of Gunion et al.

We consider the THDM of [16, 17] with the Higgs potential

$$\begin{aligned}
V(\varphi_1,\varphi_2) = {} & \lambda_1\big(\varphi_1^\dagger\varphi_1 - v_1^2\big)^2 + \lambda_2\big(\varphi_2^\dagger\varphi_2 - v_2^2\big)^2 \\
& + \lambda_3\left(\varphi_1^\dagger\varphi_1 - v_1^2 + \varphi_2^\dagger\varphi_2 - v_2^2\right)^2 \\
& + \lambda_4\left(\big(\varphi_1^\dagger\varphi_1\big)\big(\varphi_2^\dagger\varphi_2\big) - \big(\varphi_1^\dagger\varphi_2\big)\big(\varphi_2^\dagger\varphi_1\big)\right) \\
& + \lambda_5\left(\mathrm{Re}\big(\varphi_1^\dagger\varphi_2\big) - v_1 v_2\cos\xi\right)^2 \\
& + \lambda_6\left(\mathrm{Im}\big(\varphi_1^\dagger\varphi_2\big) - v_1 v_2\sin\xi\right)^2 \\
& + \lambda_7\left(\mathrm{Re}\big(\varphi_1^\dagger\varphi_2\big) - v_1 v_2\cos\xi\right) \\
& \times\left(\mathrm{Im}\big(\varphi_1^\dagger\varphi_2\big) - v_1 v_2\sin\xi\right),
\end{aligned} \tag{79}$$

which contains nine real parameters if we do not count the constant. This potential breaks the discrete symmetry

$$\varphi_1 \longrightarrow -\varphi_1\,, \qquad \varphi_2 \longrightarrow \varphi_2 \tag{80}$$

only softly, i.e. by $V_2$ terms, thus suppressing large flavour-changing neutral currents. For various restrictions on the THDM by symmetries see for instance [48]. Dropping the constant term, we put the potential into the form (29) using the relations (28). Then,

$$\begin{aligned}
\eta_{00} &= \frac{1}{4}(\lambda_1 + \lambda_2 + 4\lambda_3 + \lambda_4)\,, \\
\boldsymbol{\eta} &= \frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ \lambda_1 - \lambda_2 \end{pmatrix}, \\
E &= \frac{1}{8}\begin{pmatrix} 2(\lambda_5 - \lambda_4) & \lambda_7 & 0 \\ \lambda_7 & 2(\lambda_6 - \lambda_4) & 0 \\ 0 & 0 & 2(\lambda_1 + \lambda_2 - \lambda_4) \end{pmatrix}.
\end{aligned} \tag{81}$$

From (51) and (52) we obtain

$$\begin{aligned} f(u) &= u + \frac{1}{4}(\lambda_1 + \lambda_2 + 4\lambda_3 + \lambda_4) \\ &\quad - \frac{(\lambda_1 - \lambda_2)^2}{4(\lambda_1 + \lambda_2 - \lambda_4 - 4u)}, \\ f'(u) &= 1 - \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2 - \lambda_4 - 4u)^2}. \end{aligned} \tag{82}$$

We introduce the abbreviation

$$\kappa := \frac{1}{2}\left(\lambda_5 + \lambda_6 - \sqrt{(\lambda_5 - \lambda_6)^2 + \lambda_7^2}\right). \tag{83}$$

Applying theorem 1 to the functions $f(u)$ and $f'(u)$, we find the strong stability assertion by $V_4$, see (42), to be equivalent to the simultaneous conditions

$$\lambda_1 + \lambda_3 > 0, \qquad \lambda_2 + \lambda_3 > 0, \tag{84}$$

$$\lambda_4, \kappa > -2\lambda_3 - 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}. \tag{85}$$

In particular, if $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \kappa > 0$ these inequalities are fulfilled. They can then be rewritten as

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0, \qquad 4\lambda_5\lambda_6 > \lambda_7^2. \tag{86}$$

That means, if (86) holds, that the conditions (84) are fulfilled and the potential is stable. For the case $\lambda_7 = 0$ we can replace $\kappa$ by $\min(\lambda_5, \lambda_6)$ in the stability conditions (84), which are then in particular fulfilled if $\lambda_i > 0$ for $i = 1, \ldots, 6$.

The potential in [13] is even more specific, since it is invariant under (80). Inserting their potential parameters in (84) we reproduce the result of [13], their equation (2).

## 5 Location of stationary points

After our stability analysis in the preceding section we now determine the location of the stationary points of the potential, since among these points there are the local and global minima. To this end we define

$$\tilde{\mathbf{K}} = \begin{pmatrix} K_0 \\ \mathbf{K} \end{pmatrix}, \quad \tilde{\xi} = \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^{\mathrm{T}} \\ \boldsymbol{\eta} & E \end{pmatrix}. \tag{87}$$

In this notation the potential (29) reads

$$V = \tilde{\mathbf{K}}^{\mathrm{T}}\tilde{\boldsymbol{\xi}} + \tilde{\mathbf{K}}^{\mathrm{T}}\tilde{E}\tilde{\mathbf{K}} \tag{88}$$

and is defined on the domain

$$\tilde{\mathbf{K}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{K}} \geq 0, \qquad K_0 \geq 0, \tag{89}$$

with

$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{90}$$

For the discussion of the stationary points of $V$, we distinguish the allowed cases $\tilde{\mathbf{K}} = 0$, $K_0 > |\mathbf{K}|$ and $K_0 = |\mathbf{K}| > 0$.

The trivial configuration $\tilde{\mathbf{K}} = 0$ is a stationary point of the potential with $V = 0$, as a direct consequence of the definitions.

The stationary points of $V$ in the inner part of the domain, $K_0 > |\mathbf{K}|$, are given by

$$\tilde{E}\tilde{\mathbf{K}} = -\frac{1}{2}\tilde{\boldsymbol{\xi}} \quad \text{with} \quad \tilde{\mathbf{K}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{K}} > 0 \quad \text{and} \quad K_0 > 0. \tag{91}$$

For $\det \tilde{E} \neq 0$ we obtain the unique solution

$$\tilde{\mathbf{K}} = -\frac{1}{2}\tilde{E}^{-1}\tilde{\boldsymbol{\xi}}, \tag{92}$$

provided that

$$\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\tilde{E}^{-1}\tilde{g}\tilde{E}^{-1}\tilde{\boldsymbol{\xi}} > 0 \quad \text{and} \quad K_0 > 0, \tag{93}$$

and no solution if (93) does not hold. The Hessian matrix

$$\left(\frac{\partial^2}{\partial K_i\,\partial K_j}V\right) = 2\tilde{E}, \quad \text{where} \quad i,j = 0\ldots 3, \tag{94}$$

determines whether (92) is a local minimum, a local maximum or a saddle. In the case $\det \tilde{E} = 0$ we may have exceptional solutions of (91). In the regular case as well as in the exceptional cases, the existence of a solution of (91) along with the corresponding values of the potential are not affected by the transformation of parameters (35).

The stationary points of $V$ on the domain boundary $K_0 = |\mathbf{K}| > 0$ are stationary points of the function

$$\tilde{F}(\tilde{\mathbf{K}}, u) := V - u\tilde{\mathbf{K}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{K}}, \tag{95}$$

where $u$ is a Lagrange multiplier. The relevant stationary points of $\tilde{F}$ are given by

$$(\tilde{E} - u\tilde{g})\tilde{\mathbf{K}} = -\frac{1}{2}\tilde{\boldsymbol{\xi}} \quad \text{with } \tilde{\mathbf{K}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{K}} = 0 \text{ and } K_0 > 0. \tag{96}$$

For regular values of $u$ with $\det(\tilde{E} - u\tilde{g}) \neq 0$ we obtain

$$\tilde{\mathbf{K}}(u) = -\frac{1}{2}\left(\tilde{E} - u\tilde{g}\right)^{-1}\tilde{\boldsymbol{\xi}}. \tag{97}$$

The Lagrange multiplier is determined from the constraints in (96) by inserting (97):

$$\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\left(\tilde{E} - u\tilde{g}\right)^{-1}\tilde{g}\left(\tilde{E} - u\tilde{g}\right)^{-1}\tilde{\boldsymbol{\xi}} = 0 \quad \text{and} \quad K_0 > 0. \tag{98}$$

There may be up to four values $u = \tilde{\mu}_a$ with $a = 1, \ldots, 4$ for which $\det(\tilde{E} - u\tilde{g}) = 0$. Depending on the potential some or all of them may lead to exceptional solutions of (96). Note that for the regular as well as for the exceptional cases, the Lagrange multipliers $u$ and the value of the potential belonging to solutions $(u, \tilde{\mathbf{K}})$ of (96) are, similar to above, invariant under the transformations (35).

For *any* stationary point the potential is given by

$$V|_{\mathrm{stat}} = \frac{1}{2}\tilde{\mathbf{K}}^{\mathrm{T}}\tilde{\boldsymbol{\xi}} = -\tilde{\mathbf{K}}^{\mathrm{T}}\tilde{E}\tilde{\mathbf{K}}. \tag{99}$$

Suppose now that the weak stability condition (41) holds. Then (99) gives for non-trivial stationary points where $\tilde{\mathbf{K}} \neq 0$:

$$V|_{\text{stat}} < 0\,, \tag{100}$$

since the cases $V_4 < 0$ and $V_4 = V_2 = 0$ are excluded by the stability condition.

Similarly to the stability analysis in Sect. 4 we can use a unified description for the regular stationary points of $V$ with $K_0 > 0$ for both $|\mathbf{K}| < K_0$ and $|\mathbf{K}| = K_0$ defining the function

$$\tilde{f}(u) := \tilde{F}\big(\tilde{\mathbf{K}}(u), u\big)\,, \tag{101}$$

where $\tilde{\mathbf{K}}(u)$ is the solution (97). It follows that

$$\tilde{f}(u) = -\frac{1}{4}\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\big(\tilde{E} - u\tilde{g}\big)^{-1}\tilde{\boldsymbol{\xi}}\,, \tag{102}$$

$$\tilde{f}'(u) = -\frac{1}{4}\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\big(\tilde{E} - u\tilde{g}\big)^{-1}\tilde{g}\big(\tilde{E} - u\tilde{g}\big)^{-1}\tilde{\boldsymbol{\xi}}\,. \tag{103}$$

Denoting the first component of $\tilde{\mathbf{K}}(u)$ as $K_0(u)$ we summarise as follows.

**Theorem 2.** *The stationary points of the potential are given by*

(*I a*) $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(0)$ *if* $\tilde{f}'(0) < 0$, $K_0(0) > 0$ *and* $\det \tilde{E} \neq 0$,
(*I b*) *solutions* $\tilde{\mathbf{K}}$ *of* (91) *if* $\det \tilde{E} = 0$,
(*II a*) $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(u)$ *for* $u$ *with* $\det(\tilde{E} - u\tilde{g}) \neq 0$, $\tilde{f}'(u) = 0$ *and* $K_0(u) > 0$,
(*II b*) *solutions* $\tilde{\mathbf{K}}$ *of* (96) *for* $u$ *with* $\det(\tilde{E} - u\tilde{g}) = 0$,
(*III*) $\tilde{\mathbf{K}} = 0$.

In many cases, for instance if all values $\tilde{\mu}_1, \ldots, \tilde{\mu}_4$ are different, we can diagonalise the in general non-hermitian matrix $\tilde{g}\tilde{E}$ in the following way:

$$\tilde{g}\tilde{E} = \sum_{a=1}^{4} \tilde{\mu}_a \tilde{\mathbb{P}}_a\,. \tag{104}$$

Here the $\tilde{\mathbb{P}}_a$ are quasi-projectors constructed from the normalised right and left eigenvectors $\chi_a, \tilde{\chi}_a$ of $\tilde{g}\tilde{E}$. We have then $\tilde{g}\tilde{E}\,\chi_a = \tilde{\mu}_a\,\chi_a$, $\tilde{\chi}_a\,\tilde{g}\tilde{E} = \tilde{\chi}_a\,\tilde{\mu}_a$, $\tilde{\chi}_a\chi_b = \delta_{a\,b}$ and can impose as additional normalisation condition $\chi_a^\dagger\chi_a = 1$. The $\tilde{\mathbb{P}}_a$ are given by

$$\tilde{\mathbb{P}}_a = \chi_a\tilde{\chi}_a \tag{105}$$

and satisfy

$$\operatorname{tr}\tilde{\mathbb{P}}_a = 1, \qquad \tilde{\mathbb{P}}_a\tilde{\mathbb{P}}_b = \begin{cases} \tilde{\mathbb{P}}_a & \text{for } a = b\,, \\ 0 & \text{for } a \neq b\,, \end{cases} \tag{106}$$

where $a, b \in \{1, \ldots, 4\}$. In terms of the $\tilde{\mathbb{P}}_a$ (102) and (103) read

$$\tilde{f}(u) = -\frac{1}{4}\sum_{a=1}^{4} \frac{\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\tilde{\mathbb{P}}_a\,\tilde{g}\,\tilde{\boldsymbol{\xi}}}{\tilde{\mu}_a - u}\,, \tag{107}$$

$$\tilde{f}'(u) = -\frac{1}{4}\sum_{a=1}^{4} \frac{\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\tilde{\mathbb{P}}_a\,\tilde{g}\,\tilde{\boldsymbol{\xi}}}{(\tilde{\mu}_a - u)^2}\,. \tag{108}$$

Of course, $\tilde{f}(u)$ in (102) is always a meromorphic function of $u$, but in general poles of higher order than one may also occur.

## 6 Criteria for electroweak symmetry breaking

The global minimum will be among the stationary points discussed in the previous section. Here we discuss the spontaneous symmetry-breaking features of the possible classes of minima and give criteria to ensure a global minimum with the required electroweak symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$.

A global minimum at $\tilde{\mathbf{K}} = 0$ means vanishing fields for the vacuum. In this case, no symmetry is spontaneously broken. If the global minimum lies at $\tilde{\mathbf{K}} \neq 0$, the full gauge group or a subgroup is broken. We denote the vacuum expectation values, i.e. the fields at the global minimum of the potential $V$, by

$$v_i^+ := \langle\varphi_i^+\rangle\,, \qquad v_i^0 := \langle\varphi_i^0\rangle\,, \tag{109}$$

with $i = 1, 2$. In general the $v_i^+$, $v_i^0$ are complex numbers. To exhibit the consequences of electromagnetic gauge invariance we consider the matrix (25) at the global minimum, $\underline{K}|_{\min}$.

If the global minimum of $V$ occurs with $K_0 > |\mathbf{K}|$, it follows that $\det \underline{K}|_{\min} > 0$; see Sect. 3. Since we have

$$\det \underline{K}|_{\min} = \left|v_1^+\,v_2^0 - v_1^0\,v_2^+\right|^2\,, \tag{110}$$

the vectors

$$\begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix}, \qquad \begin{pmatrix} v_1^0 \\ v_2^0 \end{pmatrix} \tag{111}$$

are linearly independent. Then there is no transformation (30) such that both $v_1^+$ and $v_2^+$ become zero. This means that the full gauge group $SU(2)_L \times U(1)_Y$ is broken. We can also show this using the methods explained in Appendix A; see (A.23).

In the case that the global minimum of $V$ features $K_0 = |\mathbf{K}| > 0$, the rank of the matrix $\underline{K}|_{\min}$ is 1 and the vectors (111) are linearly dependent. After performing a $SU(2)_L \times U(1)_Y$ transformation we achieve

$$\begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} = 0\,, \tag{112}$$

$$\begin{pmatrix} v_1^0 \\ v_2^0 \end{pmatrix} = \begin{pmatrix} |v_1^0| \\ |v_2^0|\,\mathrm{e}^{\mathrm{i}\zeta} \end{pmatrix} \neq 0, \qquad \zeta \in \mathbb{R}\,, \tag{113}$$

and we identify the unbroken $U(1)$ gauge group with the electromagnetic one. By a transformation (30), namely

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta\,\mathrm{e}^{-\mathrm{i}\zeta} \\ -\sin\beta\,\mathrm{e}^{\mathrm{i}\zeta} & \cos\beta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{114}$$

with $\beta$ fulfilling $|v_1^0| \sin\beta = |v_2^0| \cos\beta$, we can arrange that

$$\begin{pmatrix} v_1'^{+} \\ v_2'^{+} \end{pmatrix} = 0\,, \tag{115}$$

$$\begin{pmatrix} v_1'^{0} \\ v_2'^{0} \end{pmatrix} = \begin{pmatrix} v_0/\sqrt{2} \\ 0 \end{pmatrix}, \qquad v_0 > 0\,. \tag{116}$$

This can also be derived from the results in Appendix A; see (A.16) et seq. In (116) $v_0$ is the usual Higgs-field vacuum expectation value, $v_0 \approx 246$ GeV (see for instance [46]).

Now, we want to derive conditions for the parameters in the general potential (29), which lead to the required EWSB by a global minimum with $K_0 = |\mathbf{K}| > 0$. In the following, we assume the potential to be stable. If we consider parameters fulfilling $\xi_0 \geq |\boldsymbol{\xi}|$ this immediately implies $J_2(\mathbf{k}) \geq 0$ and hence from (41) $V > 0$ for all $\tilde{\mathbf{K}} \neq 0$. Therefore for these parameters the global minimum is at $\tilde{\mathbf{K}} = 0$. Thus we arrive at the requirement

$$\xi_0 < |\boldsymbol{\xi}|\,. \tag{117}$$

Here we obtain

$$\left.\frac{\partial V}{\partial K_0}\right|_{\substack{\mathbf{k}\text{ fixed,}\\ K_0=0}} = \xi_0 + \boldsymbol{\xi}^{\mathrm{T}}\mathbf{k} < 0 \tag{118}$$

for some $\mathbf{k}$, i.e. the global minimum of $V$ lies at $\tilde{\mathbf{K}} \neq 0$.

Addressing the non-trivial cases, suppose that the two points

$$\tilde{\mathbf{p}} = \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix}, \qquad \tilde{\mathbf{q}} = \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}, \tag{119}$$

with $p_0 \geq |\mathbf{p}|$ and $q_0 \geq |\mathbf{q}|$ are stationary points of $V$; that is, each of them is either a solution of (91), or, together with an appropriate Lagrange multiplier $u_p$ or $u_q$ for $\tilde{\mathbf{p}}$ or $\tilde{\mathbf{q}}$, respectively, a solution of (96).

Firstly, consider $p_0 = |\mathbf{p}|$. From (88) and (96) we have

$$\left.\frac{\partial V}{\partial K_0}\right|_{\substack{\mathbf{K}\text{ fixed,}\\ \tilde{\mathbf{K}}=\tilde{\mathbf{p}}}} = \xi_0 + 2(\tilde{E}\,\tilde{\mathbf{p}})_0 = 2u_p\, p_0\,. \tag{120}$$

If $u_p < 0$, there are points $\tilde{\mathbf{K}}$ with $K_0 > p_0$, $\mathbf{K} = \mathbf{p}$ and lower potential in the neighbourhood of $\tilde{\mathbf{p}}$, which therefore cannot be a minimum. We conclude that in a theory with the required EWSB the global minimum (which needs to be on the light cone) must have a Lagrange multiplier $u_0 \geq 0$. As we shall show in Sect. 7, the case $u_0 = 0$ leads to zero mass for the physical charged Higgs field at tree level. This is unacceptable from a phenomenological point of view if we disregard the possibility of very large radiative corrections. Therefore, we find as condition for an acceptable theory

$$u_0 > 0 \tag{121}$$

for the Lagrange multiplier $u_0$ of the global minimum. Secondly, for $p_0 = |\mathbf{p}|$ and $q_0 = |\mathbf{q}|$ we have from (99) and (96)

$$\begin{aligned} V(\tilde{\mathbf{p}}) - V(\tilde{\mathbf{q}}) &= \frac{1}{2}\tilde{\mathbf{p}}^{\mathrm{T}}\tilde{\boldsymbol{\xi}} - \frac{1}{2}\tilde{\mathbf{q}}^{\mathrm{T}}\tilde{\boldsymbol{\xi}} \\ &= \tilde{\mathbf{p}}^{\mathrm{T}}(u_q\,\tilde{g} - \tilde{E})\tilde{\mathbf{q}} - \tilde{\mathbf{q}}^{\mathrm{T}}(u_p\,\tilde{g} - \tilde{E})\tilde{\mathbf{p}} \\ &= (u_q - u_p)\,\tilde{\mathbf{p}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{q}}\,. \end{aligned} \tag{122}$$

Since $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ are vectors on the forward light cone, $\tilde{\mathbf{p}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{q}}$ is always non-negative and zero only for $\tilde{\mathbf{p}}$ parallel to $\tilde{\mathbf{q}}$. Furthermore, the case that two different $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$ are parallel cannot occur, since then (122) requires $V(\tilde{\mathbf{p}}) = V(\tilde{\mathbf{q}})$, while (99) and (100) imply $V(\tilde{\mathbf{p}}) \neq V(\tilde{\mathbf{q}})$ for that case. Therefore we conclude

$$u_p > u_q \Longleftrightarrow V(\tilde{\mathbf{p}}) < V(\tilde{\mathbf{q}})\,. \tag{123}$$

Assuming $p_0 = |\mathbf{p}|$ and $q_0 > |\mathbf{q}|$ we get from (99) and (91)

$$V(\tilde{\mathbf{p}}) - V(\tilde{\mathbf{q}}) = -u_p\,\tilde{\mathbf{p}}^{\mathrm{T}}\tilde{g}\tilde{\mathbf{q}} \tag{124}$$

and

$$V(\tilde{\mathbf{p}}) - V(\tilde{\mathbf{q}}) = (\tilde{\mathbf{p}} - \tilde{\mathbf{q}})^{\mathrm{T}}\,\tilde{E}\,(\tilde{\mathbf{p}} - \tilde{\mathbf{q}})\,. \tag{125}$$

The first equation implies in particular that a stationary point on the domain boundary with positive Lagrange multiplier will have a lower potential than any stationary point with $K_0 > \mathbf{K}$. From the second equation follows in this case that $\tilde{E}$ has a negative eigenvalue. Since for the stationary point $\tilde{\mathbf{q}}$ in the interior of the light cone the Hessian matrix is $2\tilde{E}$ (see (94)), we see that $\tilde{\mathbf{q}}$ cannot be a local minimum. This result and the hierarchies of the stationary points derived above agree with [20]. We summarise as follows.

**Theorem 3.** *A global minimum with the spontaneous electroweak symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{em}$ and absence of zero mass physical charged Higgs bosons*

(*I*) *requires $\xi_0 < |\boldsymbol{\xi}|$,*
(*II*) *is given and guaranteed by the stationary point of the classes (IIa) or (IIb) of theorem 2 with the largest Lagrange multiplier $u_0 > 0$.*

We remark that for two different stationary points in the inner part of the domain or with $u = 0$ on its boundary, any linear combination of them with $K_0 \geq |\mathbf{K}|$ is a stationary point as well. These points therefore belong to one connected set of degenerate stationary points. Stability requires that this set contains points with $K_0 > |\mathbf{K}|$ and is bounded by points with $K_0 = |\mathbf{K}|$. If interpreted geometrically, this degenerate set is a line segment, ellipsoidal area or volume. Together with the arguments above we find the following *mutually exclusive* possibilities for local minima, expressed in terms of the gauge invariant functions: one or multiple solutions with the required EWSB ($K_0 = |\mathbf{K}|$) *or* the aforementioned degenerate set of solutions ($K_0 \geq |\mathbf{K}|$) *or* one charge breaking solution ($K_0 > |\mathbf{K}|$) *or* the trivial solution ($\tilde{\mathbf{K}} = 0$).

## 7 Potential after electroweak symmetry breaking

We assume a stable potential which leads to the desired symmetry-breaking pattern as discussed in the previous sections and derive consequences for the resulting physical fields in the following. We choose a unitary gauge and the basis for the scalar fields such that for the vacuum expectation values relations (115) and (116) hold, and furthermore the fields satisfy

$$\begin{aligned} \varphi_1^+(x) &= 0\,, && (126)\\ \operatorname{Im}\varphi_1^0(x) &= 0\,, && (127)\\ \operatorname{Re}\varphi_1^0(x) &\geq 0\,. && (128)\end{aligned}$$

We introduce as usual a shifted Higgs field

$$\rho'(x) := \sqrt{2}\,\operatorname{Re}\varphi_1^0(x) - v_0\,. \tag{129}$$

Then the two Higgs doublets are

$$\varphi_1(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_0 + \rho'(x)\end{pmatrix}, \quad \varphi_2(x) = \begin{pmatrix}\varphi_2^+(x) \\ \varphi_2^0(x)\end{pmatrix}. \tag{130}$$

In addition to $\rho'$ there are two more neutral Higgs fields:

$$h' := \sqrt{2}\,\operatorname{Re}\varphi_2^0, \qquad h'' := \sqrt{2}\,\operatorname{Im}\varphi_2^0\,, \tag{131}$$

and the charged fields

$$H^+ := \varphi_2^+\,, \qquad H^- := \left(H^+\right)^*\,. \tag{132}$$

It is convenient to decompose $\tilde{\mathbf{K}}$ according to the power of the *physical* fields they contain:

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_{\{0\}} + \tilde{\mathbf{K}}_{\{1\}} + \tilde{\mathbf{K}}_{\{2\}}\,, \tag{133}$$

with

$$\tilde{\mathbf{K}}_{\{0\}} = \begin{pmatrix} v_0^2/2 \\ 0 \\ 0 \\ v_0^2/2\end{pmatrix}, \qquad \tilde{\mathbf{K}}_{\{1\}} = v_0\begin{pmatrix}\rho' \\ h_0' \\ h_0'' \\ \rho'\end{pmatrix}, \tag{134}$$

$$\tilde{\mathbf{K}}_{\{2\}} = \frac{1}{2}\begin{pmatrix} \rho'^2 + 2H_-H_+ + h'^2 + h''^2 \\ 2\rho' h' \\ 2\rho' h'' \\ \rho'^2 - 2H_-H_+ - h'^2 - h''^2 \end{pmatrix}. \tag{135}$$

By $u_0$ we denote again the Lagrange multiplier corresponding to the global minimum of $V$. From (96) we have

$$\tilde{E}\tilde{\mathbf{K}}_{\{0\}} = u_0\,\tilde{g}\tilde{\mathbf{K}}_{\{0\}} - \frac{1}{2}\boldsymbol{\xi}\,. \tag{136}$$

From the explicit expressions (134) and (135) we further have

$$\tilde{\mathbf{K}}_{\{0\}}^{\mathrm{T}}\,\tilde{g}\,\tilde{\mathbf{K}}_{\{0\}} = 0\,, \quad \tilde{\mathbf{K}}_{\{0\}}^{\mathrm{T}}\,\tilde{g}\,\tilde{\mathbf{K}}_{\{1\}} = 0\,. \tag{137}$$

Using (133) to (137) we obtain for the potential (88)

$$V = V_{\{0\}} + V_{\{2\}} + V_{\{3\}} + V_{\{4\}}\,, \tag{138}$$

where $V_{\{k\}}$ are the terms of $k$th order in the physical Higgs fields

$$\begin{aligned} V_{\{0\}} &= (\xi_0 + \xi_3)\, v_0^2/4\,, && (139)\\ V_{\{2\}} &= \tilde{\mathbf{K}}_{\{1\}}^{\mathrm{T}}\,\tilde{E}\,\tilde{\mathbf{K}}_{\{1\}} + 2\,u_0\,\tilde{\mathbf{K}}_{\{0\}}^{\mathrm{T}}\,\tilde{g}\,\tilde{\mathbf{K}}_{\{2\}}\,, && (140)\\ V_{\{3\}} &= 2\,\tilde{\mathbf{K}}_{\{1\}}^{\mathrm{T}}\,\tilde{E}\,\tilde{\mathbf{K}}_{\{2\}}\,, && (141)\\ V_{\{4\}} &= \tilde{\mathbf{K}}_{\{2\}}^{\mathrm{T}}\,\tilde{E}\,\tilde{\mathbf{K}}_{\{2\}}\,. && (142)\end{aligned}$$

The second order terms (140) determine the masses of the physical Higgs fields:

$$V_{\{2\}} = \frac{1}{2}(\rho', h', h'')\mathcal{M}^2_{\mathrm{neutral}}\begin{pmatrix}\rho' \\ h' \\ h''\end{pmatrix} + m^2_{H^\pm} H^+ H^- \tag{143}$$

with

$$\mathcal{M}^2_{\mathrm{neutral}} = 2\begin{pmatrix} -\xi_0 - \xi_3 & -\xi_1 & -\xi_2 \\ -\xi_1 & v_0^2\,(u_0 + \eta_{11}) & v_0^2\,\eta_{12} \\ -\xi_2 & v_0^2\,\eta_{12} & v_0^2\,(u_0 + \eta_{22}) \end{pmatrix}, \tag{144}$$

$$m^2_{H^\pm} = 2\,u_0\,v_0^2\,. \tag{145}$$

Note that the condition $u_0 > 0$ corresponds to the positivity of the charged Higgs mass squared at tree level. This result was already mentioned in Sect. 6. Generically the mass terms (143) contain seven real parameters. From (144) and (145) we see that all seven parameters are in general independent in this model.

## 8 Examples

Here we apply the general considerations of Sect. 4 to Sect. 7 to specific models.

### 8.1 MSSM Higgs potential

In this subsection, we consider the MSSM Higgs potential and reproduce the well-known results for its stability, symmetry breaking and mass spectrum (see e.g. [33] and references therein), employing the method described in the previous sections. In the notation of [64] the MSSM Higgs potential is

$$V = V_D + V_F + V_{\mathrm{soft}}\,, \tag{146}$$

with

$$\begin{aligned} V_D &= \frac{1}{8}(g^2 + g'^2)\left(H_1^\dagger H_1 - H_2^\dagger H_2\right)^2 + \frac{1}{2}g^2\left|H_1^\dagger H_2\right|^2, \\ V_F &= |\mu|^2\left(H_1^\dagger H_1 + H_2^\dagger H_2\right), \\ V_{\mathrm{soft}} &= m^2_{H_1}\,H_1^\dagger H_1 + m^2_{H_2}\,H_2^\dagger H_2 - \left(m_3^2\,H_1^{\mathrm{T}}\epsilon H_2 + \mathrm{h.c.}\right), \end{aligned} \tag{147}$$

where $H_1$ and $H_2$ are Higgs doublets with weak hypercharges $y=-1/2$ and $y=+1/2$, respectively, $m^2_{H_1}, m^2_{H_2} \in \mathbb{R}$, $m_3^2 \in \mathbb{C}$ and $|\mu|^2 \in \mathbb{R}_0^+$ are parameters of dimension mass squared. Substituting $H_1$ and $H_2$ by doublets $\varphi_1, \varphi_2$ with the same weak hypercharge $y=+1/2$ according to (22) and using the relations (28), we can put the potential in the form (29). The parameters are

$$\eta_{00} = \frac{1}{8} g^2 , \quad \boldsymbol{\eta} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \quad E = \frac{1}{8} \begin{pmatrix} -g^2 & 0 & 0 \\ 0 & -g^2 & 0 \\ 0 & 0 & g'^2 \end{pmatrix} \tag{148}$$

for $V_4 = V_D$ and

$$\xi_0 = |\mu|^2 + \frac{1}{2} \left( m^2_{H_1} + m^2_{H_2} \right) , \quad \boldsymbol{\xi} = \begin{pmatrix} -\operatorname{Re}(m_3^2), \\ \operatorname{Im}(m_3^2), \\ \frac{1}{2} \left( m^2_{H_1} - m^2_{H_2} \right) \end{pmatrix} \tag{149}$$

for $V_2 = V_F + V_{soft}$.

We determine the stability of the potential by employing theorem 1. The functions $f(u)$ (51) and $f'(u)$ (52) for the MSSM are

$$f(u) = u + \frac{1}{8} g^2 , \tag{150}$$

$$f'(u) = 1 . \tag{151}$$

The set $I$ (70) is given here by $u=0$ and the eigenvalues of $E$ (148),

$$I = \left\{ u_1 = 0 , \; u_2 = -\frac{1}{8} g^2 , \; u_3 = \frac{1}{8} g'^2 \right\} . \tag{152}$$

We find for the stationary points of $J_4$ with $u_i = u_1, u_3$ the values $J_4(\mathbf{k})|_{\text{stat}} = f(u_i) > 0$ but for those with $u_2$ the value $J_4(\mathbf{k})|_{\text{stat}} = f(u_2) = 0$. Explicitly, the stationary points of $J_4$ with $u_2$ are

$$\mathbf{k} = (\cos\phi, \sin\phi, 0)^{\mathrm{T}} , \; \phi \in \mathbb{R} , \text{ with } J_4(\mathbf{k}) = 0 . \tag{153}$$

They are known as the "D-flat" directions, since they have $V_D = 0$. For the MSSM, they prevent the stability assertion by the quartic terms alone. For the stability to be guaranteed by $V_2 > 0$ in these directions, theorem 1 gives as condition (see (78) and (72)) the inequality

$$g(u_2) - |\boldsymbol{\xi}_\perp(u_2)| \sqrt{f'(u_2)} = \xi_0 - \sqrt{\xi_1^2 + \xi_2^2} > 0 . \tag{154}$$

Inserting (149) we get

$$\left| m_3^2 \right| < |\mu|^2 + \frac{1}{2} \left( m^2_{H_1} + m^2_{H_2} \right) \tag{155}$$

as the necessary and sufficient condition for the stability of the MSSM potential in the sense of (41).

For the global minimum to be non-trivial, criterion (I) of theorem 3 gives $\xi_0 < \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$, or equivalently

$$|\mu|^2 + \frac{1}{2} \left( m^2_{H_1} + m^2_{H_2} \right) < \sqrt{ |m_3^2|^2 + \frac{1}{4} \left( m^2_{H_1} - m^2_{H_2} \right)^2 } \tag{156}$$

as a necessary and sufficient condition. We consider the acceptable global minimum candidates given by the classes (IIa) and (IIb) of theorem 2. The conditions (155) and (156) prevent exceptional solutions. The regular solutions are determined by the functions

$$\tilde{f}(u) = -\frac{1}{4} \left( \frac{\xi_0^2 - \xi_1^2 - \xi_2^2}{\frac{1}{8} g^2 - u} - \frac{\xi_3^2}{-\frac{1}{8} g'^2 - u} \right) , \tag{157}$$

$$\tilde{f}'(u) = -\frac{1}{4} \left( \frac{\xi_0^2 - \xi_1^2 - \xi_2^2}{\left( \frac{1}{8} g^2 - u \right)^2} - \frac{\xi_3^2}{\left( -\frac{1}{8} g'^2 - u \right)^2} \right) , \tag{158}$$

$$K_0(u) = -\frac{1}{2} \frac{\xi_0}{\frac{1}{8} g^2 - u} , \tag{159}$$

where we omitted the insertions (149) for a compact notation. Employing again the conditions (155) and (156) we find the following.

The function $\tilde{f}'(u)$ always has two zeros and those zeros imply values of $K_0(u)$ with opposite signs. The physical solution with $K_0(u) > 0$ has the Lagrange multiplier

$$u_0 = \frac{1}{8} \frac{|\xi_3| g^2 + \sqrt{\xi_0^2 - \xi_1^2 - \xi_2^2}\, g'^2}{|\xi_3| - \sqrt{\xi_0^2 - \xi_1^2 - \xi_2^2}} , \tag{160}$$

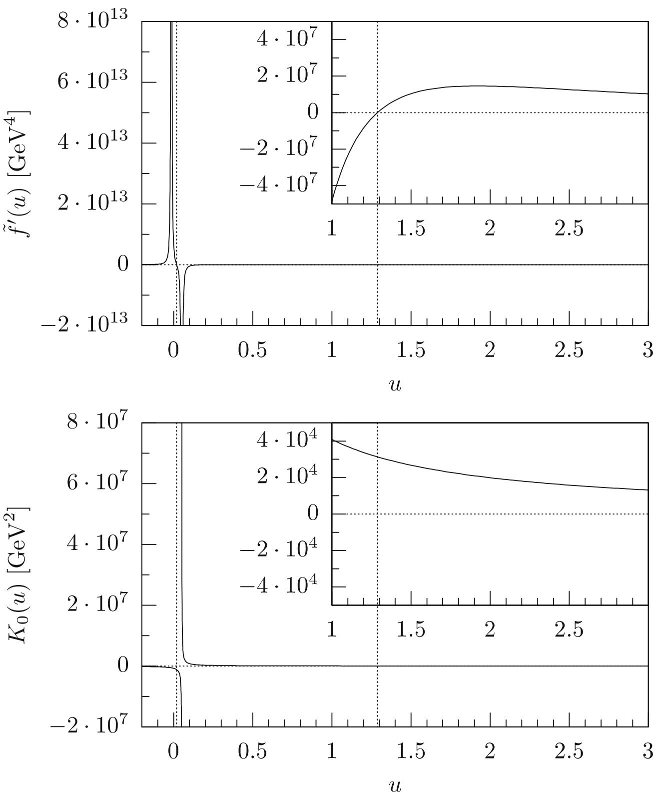


**Fig. 2.** The global minimum determining functions $\tilde{f}'(u)$ and $K_0(u)$ for the MSSM, see (158) and (159), with $|\mu|^2 + m^2_{Hd} = 157\,486\ \text{GeV}^2$, $|\mu|^2 + m^2_{Hu} = -2541\ \text{GeV}^2$, $|m_3^2| = 15\,341\ \text{GeV}^2$. The *small boxes* show the functions with enhanced ordinate resolution in the region around the physically relevant zero of $\tilde{f}'(u)$

which is positive. Figure 2 shows the functions $\tilde{f}'(u)$, $K_0(u)$ for an example set of parameters (corresponding to the SPS1a scenario [65] at the tree level). We conclude that (155) and (156) guarantee the existence of the stationary point $\tilde{\mathbf{K}}(u_0)$, which fulfils criterion (II) of theorem 3 and therefore is the global minimum with the required EWSB pattern. Moreover, there are no other local minima.

Note from (148) and (149) that it is always possible to choose a basis with $\xi_1 = -|m_3^2|$, $\xi_2 = 0$ without affecting any other parameters. We further choose a gauge where (112) and (113) hold, and perform the rotation (114) with

$$\tan\beta = \sqrt{\frac{\xi_0|\xi_3| + \sqrt{\xi_0^2 - \xi_1^2 - \xi_2^2}\,\xi_3}{\xi_0|\xi_3| - \sqrt{\xi_0^2 - \xi_1^2 - \xi_2^2}\,\xi_3}} \tag{161}$$

into a basis of the form (115) and (116), which has the new parameters

$$\boldsymbol{\xi}' = \begin{pmatrix} -c_{2\beta}|m_3^2| - s_{2\beta}\,\frac{1}{2}\left(m_{H_1}^2 - m_{H_2}^2\right) \\ 0 \\ -s_{2\beta}|m_3^2| + c_{2\beta}\,\frac{1}{2}\left(m_{H_1}^2 - m_{H_2}^2\right) \end{pmatrix}, \tag{162}$$

$$E' = \frac{1}{8}\begin{pmatrix} -g^2 + s_{2\beta}^2\,\bar{g}^2 & 0 & -\frac{1}{2}s_{4\beta}\,\bar{g}^2 \\ 0 & -g^2 & 0 \\ -\frac{1}{2}s_{4\beta}\,\bar{g}^2 & 0 & -g^2 + c_{2\beta}^2\,\bar{g}^2 \end{pmatrix}, \tag{163}$$

with the abbreviations $\bar{g}^2 := g^2 + g'^2$ and $s_{2\beta} := \sin 2\beta$ etc. We insert the expressions into the formulae of Sect. 7 and use

$$m_W^2 := \left(\frac{1}{2} g v_0\right)^2, \qquad m_Z^2 := \left(\frac{1}{2}\bar{g} v_0\right)^2, \tag{164}$$

with $v_0 = \sqrt{2K_0(u_0)}$. According to (144), the fact that $\xi_2' = \eta_{12}' = 0$ implies tree-level $CP$ conservation within the Higgs sector of the MSSM. We obtain the mass squares

$$m_{A^0}^2 = 2v_0^2\left(\eta_{22}' + u_0\right), \quad m_{H^\pm}^2 = m_{A^0}^2 + m_W^2 \tag{165}$$

for the pseudoscalar boson $A^0 := h''$ and the charged bosons $H^\pm$, which are already mass eigenstates. The non-diagonal part of the neutral mass matrix is

$$\mathcal{M}^2\big|_{\substack{\text{neutral,}\\ CP\text{ even}}} = \begin{pmatrix} c_{2\beta}^2\,m_Z^2 & -\frac{1}{2}s_{4\beta}\,m_Z^2 \\ -\frac{1}{2}s_{4\beta}\,m_Z^2 & m_{A^0}^2 + s_{2\beta}^2 m_Z^2 \end{pmatrix} \tag{166}$$

in the basis $(\rho, h')$. Its diagonalisation leads to the mass squares

$$m_{h^0,H^0}^2 = \frac{1}{2}\bigg(m_{A^0}^2 + m_Z^2 \mp \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4\,c_{2\beta}^2\,m_{A^0}^2 m_Z^2}\bigg) \tag{167}$$

for the mass eigenstates $h^0, H^0$. They are obtained from

$$\begin{pmatrix} H^0 \\ h^0 \end{pmatrix} = \begin{pmatrix} \cos\alpha' & \sin\alpha' \\ -\sin\alpha' & \cos\alpha' \end{pmatrix}\begin{pmatrix} \rho \\ h' \end{pmatrix}, \tag{168}$$

with the mixing angle $\alpha'$ determined by

$$\cos 2\alpha' = -\frac{m_{A^0}^2 - c_{4\beta}\,m_Z^2}{m_{H^0}^2 - m_{h^0}^2}, \qquad \sin 2\alpha' = -\frac{s_{4\beta}\,m_Z^2}{m_{H^0}^2 - m_{h^0}^2}. \tag{169}$$

Performing the rotation (114) on the complete doublets, as described above, leads to states $\rho, h'$ with simple couplings to the gauge bosons, e.g. vanishing $ZZh'$ and $WWh'$ couplings at tree level. However, usually the real parts of the neutral doublet components are excluded from that rotation. Applying the inverse of the rotation (114) to only the neutral components $(\rho, h')$ gives $\sqrt{2}(\operatorname{Re} H_1^1, \operatorname{Re} H_2^2)$. The mass matrix in this basis is diagonalised analogously to (168), where $\alpha'$ is replaced by the mixing angle $\alpha$ with

$$\cos 2\alpha = -\cos 2\beta\,\frac{m_{A^0}^2 - m_Z^2}{m_{H^0}^2 - m_{h^0}^2}, \qquad \sin 2\alpha = -\sin 2\beta\,\frac{m_{A^0}^2 + m_Z^2}{m_{H^0}^2 - m_{h^0}^2}, \tag{170}$$

which is the well-known result.

## 8.2 Stationary points for THDM of Gunion et al.

We continue the discussion of the potential from [16, 17], for which we derived the stability conditions in Sect. 4; see (79) et seq. Note that we consider the shifted potential according to $V(\tilde{\mathbf{K}} = 0) = 0$. The parameters of $V_4$ are given by $\eta_{00}$, $\boldsymbol{\eta}$ and $E$ as in (81), while we have

$$\xi_0 = -\lambda_1 v_1^2 - \lambda_2 v_2^2 - 2\lambda_3\left(v_1^2 + v_2^2\right), \qquad \boldsymbol{\xi} = \begin{pmatrix} -v_1 v_2\left(\lambda_5\cos\xi + \frac{\lambda_7}{2}\sin\xi\right) \\ -v_1 v_2\left(\lambda_6\sin\xi + \frac{\lambda_7}{2}\cos\xi\right) \\ -\lambda_1 v_1^2 + \lambda_2 v_2^2 \end{pmatrix} \tag{171}$$

for the parameters of $V_2$. Here, $v_1, v_2$ and $\xi$ denote the parameters of the potential (79), irrespective of their meaning for the vacuum expectation values. The function $\tilde{f}'(u)$ constructed for this potential according to (103) exhibits the zero

$$\tilde{\mu} = \frac{1}{4}\lambda_4, \tag{172}$$

whose associated solution $\tilde{\mathbf{K}}(\tilde{\mu})$ is always a stationary point. It can be represented by the field configuration

$$\varphi_1|_{\tilde{\mu}} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \qquad \varphi_2|_{\tilde{\mu}} = \begin{pmatrix} 0 \\ v_2\,\mathrm{e}^{\mathrm{i}\xi} \end{pmatrix}. \tag{173}$$

If all $\lambda_i > 0$ it is immediately obvious from the definition of the potential that this is the global minimum, which is furthermore non-trivial unless both $v_1, v_2$ are zero.

However, the quartic stability conditions (84) do not require the $\lambda_i$ to be positive. In the general case there can be more than one local minimum with the required EWSB, and the stationary point (173) is not necessarily the global minimum (we find parameters where (173) is only a saddle point, while another solution provides an admissible global minimum for a stable potential). The function $\tilde{f}'(u)$ may have up to five additional zeros which can lead to further regular stationary points. Also exceptional stationary points may occur for special parameter combinations. We do not find analytical expressions for the remaining zeros of $\tilde{f}'(u)$ for the general case. Instead we apply the methods described in the previous sections in a semi-analytical way and determine the zeros of $\tilde{f}'(u)$ numerically. We assure the stability conditions (84) to hold for the chosen parameter values. The stationary points with $K_0 = |\mathbf{K}| > 0$ and the largest Lagrange multiplier $u > 0$ are the required global minima. In order to classify also the other solutions, we compute $m^2_{H^\pm}$ and the eigenvalues of $\mathcal{M}^2_{\text{neutral}}$ for each stationary point, as described in the previous sections for the global minimum. A solution for which all of these values are positive is a local minimum. Mixed positive and negative values mean that the solution is a saddle point. Note that for a global minimum different from (173), the potential parameters $v_1, v_2$ and $\xi$ lose their simple meaning. For example, we find that the global minimum acquires a non-vanishing *CP* violating phase for certain parameters with $\xi = 0, \lambda_5 = \lambda_6 \neq 0$ and $\lambda_7 \neq 0$.

Figure 3 shows the potential at the stationary points with $K_0 = |\mathbf{K}| > 0$ for the particular parameter values described in the caption. This example features tree-level *CP* conservation within the Higgs sector: $\xi = \lambda_7 = 0$ leads to $\xi_2 = 0$ and, by (97) and $\lambda_4 \neq \lambda_6$, to $K_2 = 0$ (i.e. trivial phases) at the global minimum. In the basis with (115) and (116) we then have $\xi'_2 = \eta'_{12} = 0$, implying *CP* conservation by (144). Note that even with the simple parameter combinations chosen for Fig. 3, the structure of the stationary points can be non-trivial. In the example $\lambda_1$ and $\lambda_2$ are equal and varied simultaneously. For the plotted range, where $\lambda_1$ is negative, the global minimum is a regular solution and differs from (173), which is only a local minimum. For $\lambda_1 = 0$ there are two exceptional and degenerate minima. The figure shows for positive $\lambda_1$ the expected behaviour and shows that (173) becomes the global minimum, but also that for $0 < \lambda_1 < 0.0268$ a second regular local minimum exists. Two stationary points disappear for the plotted range above $\lambda_1 > 0.0268$ because there the corresponding two zeros of $\tilde{f}'(u)$ have a non-vanishing imaginary part.

## 9 Conclusions

We have analysed the scalar potential of the general THDM. In order to give an acceptable theory, this potential has to obey certain criteria. The potential should be stable; that is, bounded from below, and lead to the EWSB pattern observed in Nature. The conditions found for the stability of the potential and for EWSB are transparent and compact if the potential is written down in terms of gauge invariant field functions (26). These conditions allow for a simple application to every specific THDM. We illustrated our method in two examples, namely the MSSM potential as well as the THDM potential introduced by Gunion et al. [16, 17]. In the case of the MSSM we could easily reproduce all well-known results. For the potential of Gunion et al. we could clarify some interesting aspects of the model; see Fig. 3.

We note that the method presented here may also be extendable to general multi-Higgs-doublet models. A first step in this direction is done in Appendix B. Further, in a more detailed study it is mandatory to take quantum corrections to the Higgs potential into account. For the resulting effective Higgs potential the conditions for stability and for EWSB are then in general modified.

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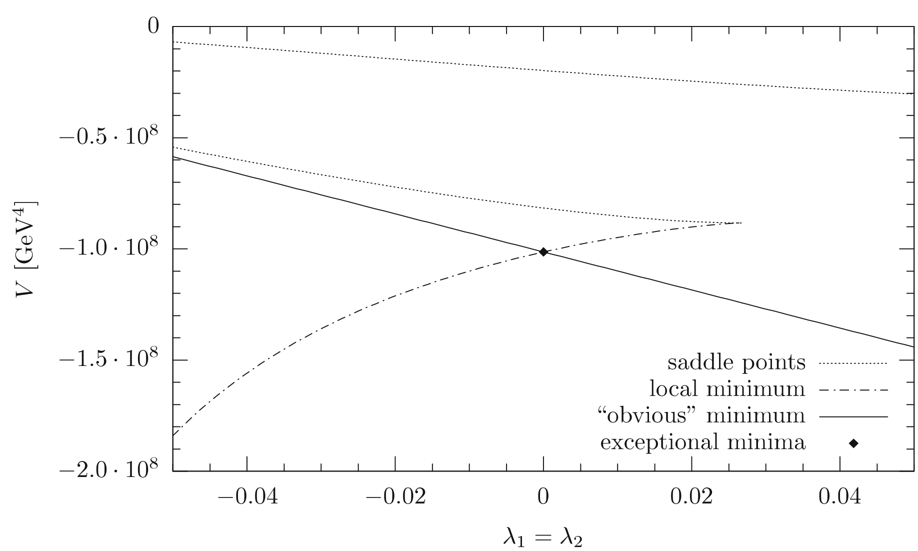


**Fig. 3.** The potential $V$ of [16, 17], shifted to $V(\tilde{\mathbf{K}} = 0) = 0$, at all stationary points with $K_0 = |\mathbf{K}| > 0$ in dependence of $\lambda_1$, where $\lambda_2 = \lambda_1$. The other parameters are $\lambda_3 = 0.1$, $\lambda_4 = 0.2$, $\lambda_5 = \lambda_6 = 0.4$, $v_1 = 30$ GeV, $v_2 = 171$ GeV, $\lambda_7 = 0$, $\xi = 0$. The *lines* represent regular stationary points, where the *solid curve* corresponds to the "obvious" solution with $\tan\beta = v_2/v_1$ and $v_0 = \sqrt{2(v_1^2 + v_2^2)}$, which is a local minimum for the chosen parameters. For $\lambda_1 = \lambda_2 = 0$ there are two degenerate exceptional minima and regular solutions only for the saddle points. Depending on $\lambda_1 = \lambda_2$, the global minimum is given by that local minimum out of the two which has the lower potential

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## Appendix A: Gauge orbits

Here we give the proof that the gauge orbits of the Higgs fields of the THDM are parametrised by four vectors $(K_0, \mathbf{K})$ satisfying (36), or equivalently, by positive-semidefinite matrices $\underline{K}$; see (25) and (26). Indeed, let us consider two Higgs doublets as in (18),

$$\varphi_i^\alpha(x), \quad i = 1, 2, \quad \alpha = +, 0\,. \tag{A.1}$$

We arrange these in a $2 \times 2$ matrix

$$\phi(x) := (\varphi_i^\alpha(x)) = \begin{pmatrix} \varphi_1^+(x) & \varphi_1^0(x) \\ \varphi_2^+(x) & \varphi_2^0(x) \end{pmatrix}. \tag{A.2}$$

Then we have from (25)

$$\underline{K}(x) = \phi(x)\phi^\dagger(x)\,. \tag{A.3}$$

The change of basis (30) means the transformation

$$\phi(x) \to \phi'(x) = U\phi(x)\,. \tag{A.4}$$

A gauge transformation from the $SU(2)_L \times U(1)_Y$ gauge group means

$$\varphi_i^\alpha(x) \to \varphi_i'^\alpha(x) = (U_G(x))_{\alpha\beta}\, \varphi_i^\beta(x)\,, \tag{A.5}$$

where

$$U_G(x) \in U(2)\,. \tag{A.6}$$

Thus, under a gauge transformation the matrix $\phi(x)$ behaves as

$$\phi(x) \to \phi'(x) = \phi(x) U_G^{\mathrm{T}}(x)\,. \tag{A.7}$$

As we discussed in Sect. 3 any matrix $\underline{K}(x)$ formed from the Higgs fields according to (25), which is equivalent to (A.3), must be positive semidefinite. Conversely, given any positive-semidefinite matrix $\underline{K}(x)$ we can diagonalise it by a $2 \times 2$ unitary transformation $W(x)$:

$$\begin{aligned} \underline{K}(x) =& W(x) \begin{pmatrix} \kappa_1(x) & 0 \\ 0 & \kappa_2(x) \end{pmatrix} W^\dagger(x)\,, \\ & W^\dagger(x) W(x) = \mathbb{1}\,. \end{aligned} \tag{A.8}$$

Since we have $\kappa_1(x) \geq 0$ and $\kappa_2(x) \geq 0$ we can set

$$\phi(x) = W(x) \begin{pmatrix} \sqrt{\kappa_1(x)} & 0 \\ 0 & \sqrt{\kappa_2(x)} \end{pmatrix} \tag{A.9}$$

and get

$$\underline{K}(x) = \phi(x)\phi^\dagger(x)\,. \tag{A.10}$$

With this we have proven the following.

- For any positive-semidefinite matrix $\underline{K}(x)$ there are Higgs fields satisfying (A.3) respectively (25).

Now suppose that we have a given positive-semidefinite matrix $\underline{K}(x)$ and two Higgs-field matrices $\phi(x)$, $\phi'(x)$, both satisfying (A.3),

$$\underline{K}(x) = \phi(x)\phi^\dagger(x) = \phi'(x)\phi'^\dagger(x)\,. \tag{A.11}$$

We want to show that $\phi'(x)$ and $\phi(x)$ are then related by a gauge transformation (A.7). We consider three cases.

1. $\underline{K}(x) = 0$. Then $\phi(x) = \phi'(x) = 0$ and (A.7) is trivially fulfilled.
2. $\underline{K}(x) > 0$, that is $\underline{K}(x)$ is positive definite. Then

$$\det \underline{K}(x) = |\det \phi(x)|^{\,2} = |\det \phi'(x)|^{\,2} > 0, \tag{A.12}$$

and both $\phi(x)$ and $\phi'(x)$ have an inverse. We set

$$\phi^{-1}(x)\phi'(x) = U_G^{\mathrm{T}}(x) \tag{A.13}$$

and find from (A.11)

$$U_G^\dagger(x) U_G(x) = \mathbb{1}\,; \tag{A.14}$$

that is, $U_G(x) \in U(2)$, and

$$\phi'(x) = \phi(x) U_G^{\mathrm{T}}(x)\,. \tag{A.15}$$

Thus $\phi'(x)$ and $\phi(x)$ satisfy (A.7), and they are related by a gauge transformation.

3. $\underline{K}(x)$ has rank 1; that is, the eigenvalues are

$$\kappa_1(x) > 0, \qquad \kappa_2(x) = 0\,. \tag{A.16}$$

With the matrix $W(x)$ diagonalising $\underline{K}(x)$ (see (A.8)) we have then from (A.11)

$$\begin{aligned} \begin{pmatrix} \kappa_1(x) & 0 \\ 0 & 0 \end{pmatrix} &= \left(W^\dagger(x)\phi(x)\right) \left(W^\dagger(x)\phi(x)\right)^\dagger \\ &= \left(W^\dagger(x)\phi'(x)\right) \left(W^\dagger(x)\phi'(x)\right)^\dagger\,. \end{aligned} \tag{A.17}$$

From this we see that

$$\begin{aligned} W^\dagger(x)\phi(x) &= \begin{pmatrix} \chi_1^+(x) & \chi_1^0(x) \\ 0 & 0 \end{pmatrix}, \\ W^\dagger(x)\phi'(x) &= \begin{pmatrix} \chi_1'^+(x) & \chi_1'^0(x) \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{A.18}$$

where

$$\chi_1^\dagger(x)\chi_1(x) = \chi_1'^\dagger(x)\chi_1'(x) = \kappa_1(x)\,, \tag{A.19}$$

with the composition of the vectors $\chi_i(x)$, $\chi_i'(x)$ defined as for the vectors $\varphi_i(x)$. Therefore we can find a matrix $U_G(x) \in U(2)$ such that

$$\chi_1'^\alpha(x) = (U_G(x))_{\alpha\beta}\, \chi_1^\beta(x)\,, \tag{A.20}$$

which implies

$$W(x)\phi'(x) = W(x)\phi(x)U_G^{\mathrm{T}}(x)\,,$$
$$\phi'(x) = \phi(x)U_G^{\mathrm{T}}(x)\,. \tag{A.21}$$

That is, $\phi'(x)$ and $\phi(x)$ are related by a gauge transformation.

With this we have completed the proof of the following statement.

**Theorem 4.** *Any two Higgs-doublet fields giving the same matrix* $\underline{K}(x)$ (25), *respectively* (A.3), *are related by a gauge transformation. The space of gauge orbits can be parametrised by four-vectors* $(K_0, \mathbf{K})$ *lying on and inside the forward light cone; see* (36).

We close this appendix with a supplementary note to Sect. 6 on the breaking of the $SU(2)_L \times U(1)_Y$ gauge group. The matrix (A.2) of vacuum expectation values is

$$\phi_{\mathrm{vac}} = \begin{pmatrix} v_1^+ & v_1^0 \\ v_2^+ & v_2^0 \end{pmatrix}. \tag{A.22}$$

If the global minimum of the potential occurs with $K_0 > |\mathbf{K}|$ the corresponding matrix $\underline{K}$ has rank 2. Then from (A.3) and (A.12) we see that also $\phi_{\mathrm{vac}}$ has rank 2. Invariance of $\phi_{\mathrm{vac}}$ under a gauge transformation (A.7),

$$\phi_{\mathrm{vac}} = \phi_{\mathrm{vac}}\, U_G^{\mathrm{T}}, \tag{A.23}$$

is then only possible for $U_G = \mathbb{1}$. That is how we see with the methods of this appendix that in this case the full gauge group $SU(2)_L \times U(1)_Y$ is broken.

## Appendix B: The case of $n$ doublets

In this appendix we generalise the methods of Sect. 3 and Appendix A to the case of $n > 2$ Higgs doublets. We consider $n$ complex Higgs-doublet fields

$$\varphi_i(x) = \begin{pmatrix} \varphi_i^+(x) \\ \varphi_i^0(x) \end{pmatrix}, \quad i = 1, \ldots, n\,. \tag{B.1}$$

All doublets are supposed to have the same weak hypercharge $y = +1/2$. In analogy to (25) we introduce the matrix

$$\underline{K}(x) = \left(\underline{K}_{ij}(x)\right) := \left(\varphi_j^\dagger(x)\varphi_i(x)\right), \tag{B.2}$$

which is now a $n \times n$ matrix. The aim is to discuss the properties of $\underline{K}(x)$. For this we introduce the $n \times n$ matrix $\phi(x)$ (compare (A.2))

$$\phi(x) := \begin{pmatrix} \varphi_1^+(x) & \varphi_1^0(x) & 0 & \ldots & 0 \\ \varphi_2^+(x) & \varphi_2^0(x) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \varphi_n^+(x) & \varphi_n^0(x) & 0 & \ldots & 0 \end{pmatrix}. \tag{B.3}$$

It is easy to see that we have

$$\underline{K}(x) = \phi(x)\,\phi^\dagger(x)\,. \tag{B.4}$$

A change of basis among the doublets means

$$\phi(x) \to \phi'(x) = U\,\phi(x) \tag{B.5}$$

with a constant matrix $U \in U(n)$,

$$U^\dagger\, U = \mathbb{1}_n\,. \tag{B.6}$$

A gauge transformation from $SU(2)_L \times U(1)_Y$ means

$$\phi(x) \to \phi'(x) = \phi(x)\,\tilde{U}_G^{\mathrm{T}}(x)\,, \tag{B.7}$$

where $\tilde{U}_G(x)$ is block-diagonal:

$$\tilde{U}_G(x) := \left(\begin{array}{c|c} U_G(x) & 0 \\ 0 & \mathbb{1}_{n-2} \end{array}\right), \tag{B.8}$$

with $U_G(x) \in U(2)$, and thus $\tilde{U}_G(x) \in U(n)$. We have then from (B.7)

$$\varphi_i'^{\alpha}(x) = (U_G(x))_{\alpha\beta}\;\varphi_i^{\beta}(x)\,, \qquad i = 1, \ldots, n\,. \tag{B.9}$$

From (B.3) and (B.4) we see that the matrix $\underline{K}(x)$ has the following properties:

- $\underline{K}(x)$ is positive semidefinite,
- $\underline{K}(x)$ has rank $\le 2$.

That is, $\underline{K}(x)$ has at most two eigenvalues $\kappa_1(x), \kappa_2(x) > 0$ and the remaining eigenvalues $\kappa_3(x), \ldots, \kappa_n(x)$ must be zero. The rank condition can be seen as follows. We denote by $\psi^+(x)$, $\psi^0(x)$ the first two column vectors of $\phi(x)$. Then we have

$$\phi(x) = \Big(\psi^+(x), \psi^0(x), 0, \ldots, 0\Big)\,, \tag{B.10}$$

$$\underline{K}(x) = \Big(\varphi_1^{+*}(x)\,\psi^+(x) + \varphi_1^{0*}(x)\,\psi^0(x), \ldots, \varphi_n^{+*}(x)\,\psi^+(x) + \varphi_n^{0*}(x)\,\psi^0(x)\Big)\,. \tag{B.11}$$

That is, at most two column vectors of $\underline{K}(x)$ are linearly independent.

Suppose now that we have a given positive-semidefinite matrix $\underline{K}(x)$ of rank $\le 2$. Then we can diagonalise $\underline{K}(x)$ and represent it as

$$\underline{K}(x) = W(x) \left(\begin{array}{cc|c} \kappa_1(x) & 0 & 0 \\ 0 & \kappa_2(x) & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) W^\dagger(x) \tag{B.12}$$

with $W(x) \in U(n)$ and $\kappa_1(x) \ge 0$, $\kappa_2(x) \ge 0$. We set now

$$\phi(x) = W(x) \left(\begin{array}{cc|c} \sqrt{\kappa_1(x)} & 0 & 0 \\ 0 & \sqrt{\kappa_2(x)} & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \tag{B.13}$$

and see easily that $\phi(x)$ is of the form (B.3) and satisfies (B.4). Thus to any positive-semidefinite matrix $\underline{K}(x)$ of rank $\leq 2$ there is at least one field configuration of the $n$ Higgs doublets such that (B.2) holds.

Suppose now that we have two field configurations; that is, two matrices $\phi(x)$ and $\phi'(x)$ of the type (B.3) such that

$$\underline{K}(x) = \phi(x)\,\phi^\dagger(x) = \phi'(x)\,\phi'^\dagger(x)\,. \tag{B.14}$$

We can diagonalise $\underline{K}(x)$ as in (B.12) and get

$$\left(\begin{array}{cc|c} \kappa_1(x) & 0 & \\ 0 & \kappa_2(x) & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) = \left(W^\dagger(x)\,\phi(x)\right)\left(W^\dagger(x)\,\phi(x)\right)^\dagger = \left(W^\dagger(x)\,\phi'(x)\right)\left(W^\dagger(x)\,\phi'(x)\right)^\dagger\,. \tag{B.15}$$

From this we see that we must have

$$W^\dagger(x)\,\phi(x) = \left(\begin{array}{cc|c} \chi_1^+(x) & \chi_1^0(x) & \\ \chi_2^+(x) & \chi_2^0(x) & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \tag{B.16}$$

$$W^\dagger(x)\,\phi'(x) = \left(\begin{array}{cc|c} \chi_1'^+(x) & \chi_1'^0(x) & \\ \chi_2'^+(x) & \chi_2'^0(x) & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \tag{B.17}$$

where

$$\begin{aligned} \chi_1^\dagger(x)\,\chi_1(x) &= \chi_1'^\dagger(x)\,\chi_1'(x) = \kappa_1(x)\,, \\ \chi_2^\dagger(x)\,\chi_2(x) &= \chi_2'^\dagger(x)\,\chi_2'(x) = \kappa_2(x)\,, \\ \chi_1^\dagger(x)\,\chi_2(x) &= \chi_1'^\dagger(x)\,\chi_2'(x) = 0\,. \end{aligned} \tag{B.18}$$

From this we conclude that we can find a matrix $U_G(x) \in U(2)$ such that

$$\chi_i'^\alpha = (U_G(x))_{\alpha\beta}\,\chi_i^\beta(x), \qquad i = 1, 2\,. \tag{B.19}$$

Inserting this $U_G(x)$ into (B.8) we get

$$W^\dagger(x)\phi'(x) = W^\dagger(x)\phi(x)\,\tilde{U}_G^T(x)\,, \tag{B.20}$$

and, since $W(x) \in U(n)$,

$$\phi'(x) = \phi(x)\tilde{U}_G^T(x)\,. \tag{B.21}$$

That is, $\phi'(x)$ and $\phi(x)$ are related by a gauge transformation. We summarise our findings in a theorem.

**Theorem 5.** *For $n$ Higgs-doublet fields of the same weak hypercharge $y = +1/2$ the matrix $\underline{K}(x) = (\varphi_j^\dagger(x)\varphi_i(x))$ is a positive-semidefinite $n \times n$ matrix of rank $\leq 2$. For any positive-semidefinite $n \times n$ matrix $\underline{K}(x)$ of rank $\leq 2$ there are Higgs fields such that* (B.2) *holds. Any two field configurations giving the same matrix $\underline{K}(x)$ are related by a $SU(2)_L \times U(1)_Y$ gauge transformation. The matrices $\underline{K}(x)$ form, therefore, the space of the gauge orbits of the $n$ Higgs-doublet fields.*

As an example we consider three Higgs doublets. The space of gauge orbits is then given by all positive-semidefinite $3 \times 3$ matrices $\underline{K}(x)$ with $\det \underline{K}(x) = 0$.

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